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Reduced tensor algebra in SU(3) Chern–Simons field theory

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Abstract

We consider the non-Abelian SU(3) Chern–Simons field theory defined in $\mathbb{R}^3$ and S^3. The reconstruction theorems for the vacuum expectation values of the Wilson line operators are proved. We give in particular the general expressions for the values of the unknot and of the Hopf link. The physical consequences of the quantization of the coupling constant are studied in detail. The structure of the resulting reduced tensor algebra is derived.

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1. Introduction

The quantum Chern–Simons (CS) field theory [1] is a nontrivial gauge theory whose action is the integral of a three-form on a three-manifold. The action does not depend on the metric that one can introduce in the manifold; this property is called general covariance. Because of general covariance, the CS field theory does not describe a set of interacting particles, as ordinary field theories do. Nevertheless, the observables of the model are highly nontrivial. These observables, which depend on the topology of knots and links in the three-manifold, represent link invariants of ambient isotopy; they also provide an algebraic representation of four-dimensional cobordism which is given in terms of three-manifold invariants. Quite remarkably, the values of the gauge invariant observables can be computed exactly with a finite number of operations. For this reason, the quan-

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tum CS field theory is the first example of a solvable nontrivial gauge theory which is defined in a generic closed, connected and orientable three-manifold.

Some general features of the quantum CS field theory have been discussed in Refs. [1,2]. The explicit solution of the non-Abelian SU(2) model in a generic three-manifold has been produced in Refs. [3,4]. When the gauge group is SU(2), the three-manifold invariants obtained in the CS field theory perfectly agree with the invariants defined by Reshetikhin and Turaev [5] by means of quantum groups. In general, the results obtained by means of the modular Hopf algebra [5] associated with quantum groups are essentially equivalent to the results of the topological field theory. This equivalence is not a coincidence but is due to the universality of the representation rings of quantum groups (when the deformation parameter is a free indeterminate) and of ordinary simple compact Lie groups. For the same reason, the several variants on three-manifold invariants, which recently appeared [6] in the literature, just represent slightly different versions of the Reshetikhin–Turaev invariant.

In the present and in a following [7] article, we shall produce the explicit solution of the quantum CS field theory when the gauge group is SU(3) in a generic three-manifold which is closed, connected and orientable. In order to illustrate the general properties of the non-Abelian CS theory, the group SU(3) is particularly convenient. The structure of the group SU(3) is rather simple but, at the same time, several peculiar features of SU(2) are absent in this case. For example, each irreducible representation of SU(3) is not necessarily real and, in the decomposition of the tensor product of two irreducible representations of SU(3), the multiplicities of the irreducible components are in general nontrivial. The surgery rules of the topological field theory are determined [3] by the structure of the reduced tensor algebra associated with the gauge group. Thus, the central issue we shall consider in this first article is the construction of the reduced tensor algebra $\mathcal{T}_{(k)}$ which, for fixed integer values of the CS coupling constant k, is associated with the group SU(3). We shall use standard cabling to prove the main reconstruction theorems of the link polynomials. We shall determine the values of the Hopf matrix, which is associated with couples of arbitrary irreducible representations of SU(3), and its properties are analyzed.

Our main purpose is to illustrate the crucial role that symmetry principles have in the topological field theory and in the definition of the associated three-manifold invariants. We will also show how symmetry arguments can be used to simplify the actual computation of these invariants.

This article is organized as follows. We shall firstly discuss the main properties of the non-Abelian SU(3) CS theory in $\mathbb{R}^3$ and we shall explain how to compute the expectation values of the Wilson line operators in closed form. We shall derive, in particular, the general expression of the unknots and of the Hopf link for arbitrary irreducible representations of SU(3). Secondly, we shall consider the CS field theory in S^3. The expectation values of the observables in S^3 can simply be obtained from the expectation values in $\mathbb{R}^3$ by adding the condition that the coupling constant is an integer. For each integer value of the coupling constant, the associated reduced tensor algebra will be constructed.

2. The action principle

The field theory model in which we are interested is defined by the action

$$S_{CS} = \frac{k}{4\pi} \int d^3x \, \epsilon^{\mu\nu\rho} \, \mathrm{Tr}(A_\mu \partial_\nu A_\rho + i\tfrac{2}{3} A_\mu A_\nu A_\rho) \,, \tag{2.1}$$

where $A_\mu = A^a_\mu T^a$ and $\{T^a\}$ are the generators of the gauge group SU(3) in the fundamental (three-dimensional) representation. The eight matrices $\{T^a\}$ are normalized as

$$\mathrm{Tr}(T^a T^b) = \tfrac{1}{2}\delta^{ab} \,. \tag{2.2}$$

With the introduction of the gauge-fixing and ghost terms, in the Landau gauge the total action for the theory defined in $\mathbb{R}^3$ is [8]

$$\begin{aligned} S = \frac{k}{4\pi} \int d^3x \, \{ & \tfrac{1}{2} \epsilon^{\mu\nu\rho} A^a_\mu \partial_\nu A^a_\rho - \tfrac{1}{6} \epsilon^{\mu\nu\rho} f^{abc} A^a_\mu A^b_\nu A^c_\rho \\ & - B^a \partial^\mu A^a_\mu + \bar{c}^a \partial^\mu (\partial_\mu c^a - f^{abc} A^b_\mu c^c) \} \,, \end{aligned} \tag{2.3}$$

where $\{f^{abc}\}$ are the structure constants of SU(3) and where the ordinary flat metric $\delta_{\mu\nu}$ of $\mathbb{R}^3$ has been used to contract the vector indices in the gauge-fixing and ghost Lagrangian terms. The total action is invariant under the BRS transformations [9]

$$\begin{aligned} &\Delta A^a_\mu = \partial_\mu c^a - f^{abc} A^b_\mu c^c, \qquad \Delta B^a = 0 \,, \\ &\Delta c^a = \tfrac{1}{2} f^{abc} c^b c^c, \qquad \Delta \bar{c}^a = B^a \,. \end{aligned} \tag{2.4}$$

The CS theory in $\mathbb{R}^3$ is renormalizable; the beta function and the anomalous dimensions of the elementary fields are vanishing [10] to all orders of perturbation theory. The only free parameter of the model is the value of the renormalized coupling constant k; we shall now fix the normalization conditions for k. Let $\Gamma = [A_\mu, B, \bar{c}, c]$ be the effective action of the theory. Since there are no gauge anomalies, one can always construct an effective action Γ which is BRS invariant. This means that, in the limit in which the regularization cutoffs are removed, Γ is BRS invariant. BRS invariance will be used to fix the normalization of the fields; indeed, as a function of the renormalized fields, Γ must satisfy

$$\left\{ \Delta A^a_\mu \frac{\delta}{\delta A^a_\mu} + \Delta B \frac{\delta}{\delta B} + \Delta c \frac{\delta}{\delta c} + \Delta \bar{c} \frac{\delta}{\delta \bar{c}} \right\} \Gamma = 0 \,. \tag{2.5}$$

The effective action admits an expansion in powers of the fields, of course; let us consider for instance the terms $\Gamma^{(2)}[A_\mu]$ and $\Gamma^{(3)}[A_\mu]$ of this expansion which are quadratic and cubic in the field A_μ,

$$\Gamma^{(2)}[A_\mu] = \int d^3x\, d^3y\, G^{\mu\nu}_{ab}(x,y) A^a_\mu(x) A^b_\nu(y) \,, \tag{2.6}$$

$$\Gamma^{(3)}[A_\mu] = \int d^3x\, d^3y\, d^3z\, H^{\mu\nu\rho}_{abc}(x,y,z) A^a_\mu(x) A^b_\nu(y) A^c_\rho(z) \,. \tag{2.7}$$

The BRS invariance (2.5) implies that the exact two-point function $G^{\mu\nu}_{ab}(x,y)$ is related to the three-point proper vertex $H^{\mu\nu\rho}_{abc}(x,y,z)$ by

$$\partial^x_\mu H^{\mu\nu\rho}_{abc}(x,y,z) = \tfrac{2}{3} f^d{}_{ab} G^{\nu\rho}_{dc}(x,z) \delta^3(x-y) \,. \tag{2.8}$$

Our normalization of the fields is fixed by eq. (2.5); equivalently, our normalization of the elementary vector field A_μ is determined by eq. (2.8). Let us now fix the normalization condition for k. As shown in Ref. [10], the exact two-point function $G^{\mu\nu}_{ab}(x,y)$ has the form

$$G^{\mu\nu}_{ab}(x,y) = Z \epsilon^{\mu\rho\nu} \partial^x_\rho \delta^3(x-y) \delta_{ab} \,, \tag{2.9}$$

where Z is a real parameter. The parameter Z which is computed in perturbation theory, is a function of the "bare" coupling constant k_0. The functional dependence of Z on k_0 can be arbitrarily modified by choosing different regularization prescriptions and/or by adding finite local counterterms at each order of the loop expansion, of course. As explained in any textbook, how Z depends on k_0 is totally irrelevant because the bare coupling constant itself has no physical (or intrinsic) meaning. The value of the coupling constant that one can measure in a laboratory is called the physical coupling constant and is determined by the complete (resummed to all orders) theory. In physics, the value of the physical coupling constant is also called [11] the renormalized coupling constant and is usually determined in terms of the effective action Γ of the theory. We shall follow the standard field theory procedure and the value of the renormalized coupling constant k of the CS theory will be fixed by

$$k = 8\pi Z \,, \tag{2.10}$$

where Z enters the exact two-point function (2.9). With the field normalization (2.8), eq. (2.10) represents the normalization condition for k.

Condition (2.10) gives a good definition of the renormalized coupling constant k because eq. (2.10) is in agreement with the classical expression (2.3) of the action. Consequently, when the correlation functions of the fields are expressed in term of k, the equations which follow from the action principle are satisfied. Vice versa, for the renormalized correlation functions the action principle is valid [11] only if the coupling constant k satisfies condition (2.10).

All the results that we shall derive are consequences of the action principle based on the functional (2.3); the expectation values of the gauge invariant observables will be expressed in terms of the physical coupling constant k. The coupling constant k of the SU(3) CS theory should not be confused with the level l of the two-dimensional $SU(3)_l$ WZNW conformal model.

A remarkable property of the quantum CS theory is that, according to eq. (2.9), the two-point function does not receive radiative corrections; therefore, the full (dressed) propagator coincides with the free one,

$$\langle A^a_\mu(x) A^b_\nu(y) \rangle = \frac{i}{k} \delta^{ab} \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} . \tag{2.11}$$

Let us recall that the function defined by the propagator, integrated along two oriented, closed and non-intersecting paths C_1 and C_2 in $\mathbb{R}^3$, must represent an invariant of ambient isotopy for a two-component link. The invariant associated with the expression (2.11) is simply the linking number [9,12] of C_1 and C_2,

$$\mathrm{lk}(C_1, C_2) = \frac{1}{4\pi} \oint_{C_1} dx^\mu \oint_{C_2} dy^\nu \, \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} . \tag{2.12}$$

3. Composite Wilson line operators

Let C be an oriented knot in $\mathbb{R}^3$ with a given irreducible representation ρ assigned to it, the classical expression of the associated Wilson line operator is

$$W(C;\rho) = \mathrm{Tr}\, P \exp\left(i \oint_C A^a_\mu(x) T^a_{(\rho)} \, dx^\mu \right) , \tag{3.1}$$

where the path-ordering is defined according to the orientation of C and $\{T^a_{(\rho)}\}$ are the generators of the gauge group in the representation ρ. The classical expression (3.1) corresponds to a composite operator; consequently, a precise definition of the Wilson line operator must be given at the quantum level. This means that one has to specify, for instance, the operative procedure which must be used, at any given order of perturbation theory, in the computation of the expectation values of this operator. The only known procedure which preserves general covariance is the framing procedure introduced in Refs. [2,8]. Therefore, we shall define the composite Wilson line operators at the quantum level by means of the framing procedure.

For each oriented knot C, we shall introduce a framing C_f. The framing C is completely determined, up to ambient isotopy, by the linking number $\mathrm{lk}(C, C_f)$ of C and C_f. When the linking number of C and C_f is vanishing, C_f is called a preferred framing for C. Given an oriented framed knot C in $\mathbb{R}^3$ and an irreducible representation ρ of the gauge group, the associated Wilson line operator $W(C; \rho)$ is well defined and is gauge invariant.

Let us now consider a framed, oriented and coloured link L in $\mathbb{R}^3$ with m components $\{C_1, C_2, ..., C_m\}$. Let the colour of each component C_i of L be specified by an irreducible representation ρ_i of the gauge group. The Wilson line operator $W(L)$ associated with L is simply the product of the Wilson operators defined for the single components

$$W(L) = W(C_1;\rho_1)\, W(C_2;\rho_2) ... W(C_m;\rho_m) \ . \tag{3.2}$$

The set of expectation values

$$E(L) = \langle W(L) \rangle |_{\mathbb{R}^3} = \frac{\langle 0 | W(L) 0 | \rangle |_{\mathbb{R}^3}}{\langle 0 | 0 \rangle |_{\mathbb{R}^3}} \ , \tag{3.3}$$

which are defined for all the possible links $\{L\}$ which are framed, oriented and coloured, is the set of the gauge invariant observables in which we are interested.

For any given link L in $\mathbb{R}^3$, the expectation value (3.3) admits an expansion in powers of $(2\pi/k)$; each term of this expansion can be computed by means of ordinary perturbation theory. In order to verify that the observable $E(L)$ is well defined, let us consider for instance the first three terms of this expansion.

The Wilson line operators (3.2) are defined for links which are contained inside some finite and closed domain of $\mathbb{R}^3$. For this kind of observables, the vacuum expectation values can be computed by means of the standard BRS quantization procedure which is based on the action (2.3). In $\mathbb{R}^3$ there are no constraints to be imposed on the value of the coupling constant k; thus, perturbation theory is well defined because the expansion parameter λ, given by

$$\lambda = (2\pi/k) \ , \tag{3.4}$$

is a free parameter and the expectation values $E(L)$ are well defined for arbitrary values of λ. This is why we chose $\mathbb{R}^3$ as starting manifold; in fact, all the perturbative aspects of the CS model refer to the theory defined in $\mathbb{R}^3$.

Note that in a generic three-manifold $\mathcal{M}$, which is closed connected and orientable, λ is no more a free parameter; gauge invariance under large gauge transformations in $\mathcal{M}$ implies [1] that k must take certain integer values. Consequently, ordinary perturbation theory in $\mathcal{M}$ is expected to be (in general) ill defined. Our way to solve the quantum CS theory in a generic manifold $\mathcal{M}$ consists of three steps. Firstly, we will solve the theory in $\mathbb{R}^3$, where ordinary perturbation theory is reliable and defines the theory unambiguously. Secondly, by taking into account the behaviour of the action under large gauge transformations, we will extend the results obtained in $\mathbb{R}^3$ to the case of the three-sphere S^3. Finally, we will use the symmetry properties of the topological theory to solve the model in a generic three-manifold $\mathcal{M}$. In this context, the relevant symmetry we will need to consider is related to twist homeomorphisms of solid tori.

Let us consider the unknot U (simple circle) in $\mathbb{R}^3$ with framing U_f and with colour given by the irreducible representation ρ of SU(3). The first three terms of the λ-expansion of the vacuum expectation value of the associated Wilson line operator $W(U, U_f; \rho)$ have been computed [8] by means of ordinary Feynman diagrams; the result is

$$\langle W(U, U_f; \rho)\rangle|_{\mathbb{R}^3} = (\dim \rho)\,[1 - i\lambda Q(\rho)\,\mathrm{lk}(U, U_f) - \tfrac{1}{2}\lambda^2 Q^2(\rho)\,(\mathrm{lk}(U, U_f))^2 - \tfrac{1}{4}\lambda^2 Q(\rho)] + O(\lambda^3)\,, \tag{3.5}$$

where $Q(\rho)$ is the value of the quadratic Casimir operator in the representation ρ,

$$T^a_{(\rho)} T^a_{(\rho)} = Q(\rho)\cdot \mathbb{1}\,. \tag{3.6}$$

Let us now consider the Hopf link in $\mathbb{R}^3$ in which the two components C_1 and C_2 are oriented as shown in Fig. 3.1. Let C_1 and C_2 have framings C_{1f} and C_{2f}, respectively. When the colours of C_1 and C_2 are given by the irreducible representations ρ and ρ' of SU(3), one finds [8]

$$\begin{aligned}\langle W(C_1, C_{1f}; \rho)\, W(C_2, C_{2f}; \rho')\rangle|_{\mathbb{R}^3} &= (\dim \rho)(\dim \rho')\\ &\times [1 - i\lambda Q(\rho)\mathrm{lk}(C_1, C_{1f}) - i\lambda Q(\rho')\mathrm{lk}(C_2, C_{2f})\\ &- \tfrac{1}{2}\lambda^2 Q^2(\rho)(\mathrm{lk}(C_1, C_{1f}))^2 - \tfrac{1}{2}\lambda^2 Q^2(\rho')(\mathrm{lk}(C_2, C_{2f}))^2\\ &- \tfrac{1}{4}\lambda^2 Q(\rho) - \tfrac{1}{4}\lambda^2 Q(\rho') - \tfrac{1}{4}\lambda^2 Q(\rho) Q(\rho')] + O(\lambda^3)\,.\end{aligned} \tag{3.7}$$

In conclusion, perturbation theory is well defined in $\mathbb{R}^3$ and the expectation values of the observables, which are defined by the framing procedure, represent ambient isotopy invariants of framed, oriented and coloured links in $\mathbb{R}^3$.

For the Abelian CS model, the series determined by perturbation theory can easily be summed and, consequently, one can produce [2] the exact expression of the expectation values of the observables. In the non-Abelian case, the computation of the contributions of higher orders in powers of λ becomes very complicated. Thus, in the non-Abelian case the direct resummation of the perturbative expansion is not practical. Nevertheless, for any assigned link L in $\mathbb{R}^3$, one can still find the exact expression of $\langle W(L)\rangle|_{\mathbb{R}^3}$. In order to do this, one has to combine [2,4] the general properties of the expectation values with the numerical information which can be obtained [13] in the Schrödinger picture. The main properties of the expectation values are encoded in the satellite relations [2,4]. The relevant numerical information, which can be obtained in the canonical for-

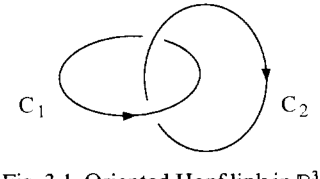

Fig. 3.1. Oriented Hopf link in $\mathbb{R}^3$.

malism, is represented by the structure [13] of the monodromy matrices or, equivalently, by the "eigenvalues" of the braiding matrix associated with an exchange of two link components.

The exact expression of $E(L) = \langle W(L) \rangle |_{\mathbb{R}^3}$ can be obtained by using the rules introduced in Ref. [2]. These rules have been derived [2] from the quantum CS field theory (with gauge group SU(N)) and have been formulated for link diagrams. In the computation of the observables, these rules summarize all the information encoded in the field theory action principle. In fact, all the results which are obtained according to these rules coincide with the results obtained, for instance, by means of ordinary perturbation theory. A detailed discussion of the derivation and of the applications of these rules can be found in Ref. [4]. In the present article, we shall firstly recall the main properties of the expectation values $\{E(L)\}$ which are consequences of the symmetries of the field theory and of the structure of the Wilson line operators. Then, we shall use standard cabling to prove the main reconstruction theorems for the observables $\{E(L)\}$ when the gauge group is $G=\mathrm{SU}(3)$.

4. Symmetries and satellites

The quantum CS field theory provides and intrinsic three-dimensional description of the link invariants. Thus, several features of $\{E(L)\}$ admit a simple physical interpretation which is a consequence of the symmetries of the field theory. In this section, we shall recall some general properties of the expectation values which will be useful for our subsequent discussion.

If the link L is the distant (disjoint) union of the links L_1 and L_2, we shall write $L=L_1 \cup L_2$. General covariance implies [2] that

$$E(L_1 \cup L_2)=E(L_1)E(L_2) . \tag{4.1}$$

Let the irreducible representation ρ be assigned to the oriented component C of the link L. Let us denote by L' the coloured link which can be obtained from L according to the following prescription; replace the representation ρ with its complex conjugate ρ^* and, simultaneously, modify the orientation of the component C. Then, from the definition of the Wilson line operators, it follows [2] that

$$E(L')=E(L) . \tag{4.2}$$

Clearly, if the representation ρ which is assigned to the component C is the trivial representation, then the associated Wilson line operator is the identity and, in this case, the link component C can simply be eliminated.

Let the finite-dimensional representation ρ be associated with the framed and oriented knot C. The Wilson line operator $W(C;\rho)$ is also well defined when ρ is not irreducible. Indeed, in this case we can decompose ρ into a direct sum of its irreducible components and, since $W(C;\rho)$ is the trace of the quantum holonomy, $W(C;\rho)$ can accordingly be written as a sum of the Wilson line operators defined for these irreducible components. Consequently, for fixed knot C, $E(C)$ can be understood as a linear function on the representation ring of the gauge group. One can extend the representation ring of the gauge group to a full algebra $\mathscr{T}$ over the complex numbers. This associative and commutative algebra is called [4,14] the tensor algebra. The structure constants of $\mathscr{T}$ are given by the multiplicities of the irreducible representations contained in the tensor product of two given representations. To be more precise, for each irreducible representation ρ of the gauge group, we shall introduce an element $\chi[\rho]\in\mathscr{T}$; the set $\{\chi[\rho]\}$ of all these elements, which are defined for all the inequivalent irreducible representations, is the set which contains the elements of the standard basis of $\mathscr{T}$. The structure constants of $\mathscr{T}$ are given by

$$\chi[\rho_1]\chi[\rho_2]=\sum_{\rho} F_{\rho_1,\rho_2,\rho}\chi[\rho] , \tag{4.3}$$

where $F_{\rho_1,\rho_2,\rho}$ is the multiplicity of the irreducible representation ρ which is contained in the decomposition of the tensor product $\rho_1\otimes\rho_2$. If $\rho\notin\rho_1\otimes\rho_2$, then the corresponding coefficient $F_{\rho_1,\rho_2,\rho}$ is is vanishing. In the standard basis of $\mathscr{T}$, the structure constants take nonnegative integer values. The symmetry properties of $\{F_{\rho_1,\rho_2,\rho}\}$ are fixed by the simple Lie algebra structure of the gauge group; namely, one has

$$F_{\rho_1,\rho_2,\rho}=F_{\rho_2,\rho_1\,\rho} , \tag{4.4}$$

$$F_{\rho_1,\rho_2,\rho}=F_{\rho_1^*,\rho_2^*,\rho^*} , \tag{4.5}$$

$$F_{\rho_1,\rho_2,\rho}=F_{\rho_1^*,\rho,\rho_2} , \tag{4.6}$$

where ρ^* is the complex conjugate of the irreducible representation ρ. Clearly, if the framed and oriented link L has n components, $E(L)$ can be understood [3,4] as a multilinear function on $\mathscr{T}^{\otimes n}$.

Let us consider an oriented framed knot C and the set $\{W(C;\rho)\}$ of the associated Wilson line operators which are defined for all the possible inequivalent irreducible representations $\{\rho\}$ of the gauge group. The set $\{W(C;\rho)\}$ is a complete set [4] of gauge invariant observables associated with the knot C. This means that, if $\mathscr{O}(C)$ is a metric-independent gauge invariant observable of the CS theory which is defined in terms of the vector fields $A^a_\mu(x)$ and $\mathscr{O}(C)$ is associated to the knot C, then $\mathscr{O}(C)$ can always be written as

$$\mathscr{O}(C)=\sum_{\rho}\xi(\rho)W(C;\rho)=\sum_{\rho}\xi(\rho)W(C;\chi[\rho]) , \tag{4.7}$$

where the $\{\xi(\rho)\}$ are numerical (complex) coefficients. With a given choice of the framing of C, the coefficients $\{\xi(\rho)\}$ are fixed and characterize the observable $\mathscr{O}(C)$. Any function which is defined on the equivalence classes of conjugate elements of a compact simple group admits a linear decomposition in terms of the characters of the group, of course. Eq. (4.7) is the analogue of this decomposition for the gauge invariant observables which are associated with the knot C.

Eq. (4.7) can be used to derive the generalized satellite relations [4] which are satisfied by $E(L)$. Let V be a solid torus standardly embedded in S^3; the oriented core of V will be denoted by K and its preferred framing by K_f. Consider now the oriented and framed component C of a link L in S^3; let C_f be the framing of C. A tubular neighbourhood N of C is a solid torus embedded in S^3 whose core is C. The two solid tori V and N are homeomorphic; we shall denote by $h^\diamond$ the homeomorphism $h^\diamond : V \to N$ which has the properties

$$h^\diamond(K) = C, \qquad h^\diamond(K_f) = C_f . \tag{4.8}$$

Up to ambient isotopy, the homeomorphism $h^\diamond$ is unique and is determined by the orientation and by the framing of the link component C.

One can imagine that the framed component C of L is the image $h^\diamond(K)$ of the framed knot $K \subset V$ under the homeomorphism $h^\diamond$. Starting from the link L, we shall now construct a new link L'; L' is obtained by replacing the component C of L with the image $h^\diamond(P)$ of a given (oriented and framed) link P in V. The link L' is called a generalized satellite of L; the link L is a companion of L' and $P \subset V$ is called the pattern link.

Suppose now that L' is a generalized satellite of L defined in terms of a given pattern link P. Furthermore, suppose that an irreducible representation of the gauge group has been assigned to each component of P. We would like to know how the expectation value $E(L')$ is connected with $E(L)$; the precise relation between $E(L')$ and $E(L)$ is called a generalized satellite relation. Let us now recall the structure of the satellite relations. Remember that L' has been obtained from L by replacing the framed component C with the image $h^\diamond(P)$ of the pattern link P. By definition, P belongs to the solid torus V and $W(P)$, which denotes the product of the Wilson line operators associated with P, represents a gauge invariant observable defined in V. Since the CS model is a topological field theory, the thickness of V is totally irrelevant. In the limit in which this thickness goes to zero, the solid torus V degenerates to a simple circle: the core K of V. Thus, $W(P)$ can be understood as a gauge invariant observable associated with the knot K. Consequently, $W(P)$ admits an expansion of the type shown in eq. (4.7),

$$W(P) = \sum_\rho \xi(\rho) W(K; \chi[\rho]) . \tag{4.9}$$

The complex coefficients $\{\xi(\rho)\}$, appearing in eq. (4.9), depend on the pattern link P and on the representations assigned to its components. The composite Wilson line operators $\{W(K, \chi[\rho])\}$ are defined for the framed knot K (which has preferred framing). Let us introduce the element χ,

$$\chi = \sum_{\rho} \xi(\rho)\chi[\rho] \,, \tag{4.10}$$

of the tensor algebra $\mathscr{T}$. Then, eq. (4.9) can simply be written as

$$W(P) = W(K;\chi) \,. \tag{4.11}$$

At this stage, from the defining conditions (4.8) of $h^{\diamond}$, if follows immediately [2,4] that

$$E(L') = E(L;\ \text{with the component } C \text{ of } L \text{ associated with } \chi) \,. \tag{4.12}$$

This equation is a consequence of two basic symmetry properties of the CS theory. Firstly, gauge invariance implies that the non-Abelian "electric" flux associated with a meridional disc of a solid torus must be conserved. Secondly, general covariance implies that this flux can always be imagined to be concentrated on a single knot which coincides with the core of the solid torus.

The satellite relation (4.12) represents one of the main properties of the expectation values $\{E(L)\}$. The ξ-coefficients, appearing in eq. (4.9), can be determined by using several different methods; some of them have been presented in Refs. [3,4].

Let us consider a particular example of satellite relation which will be used in the following sections. We shall consider a particular pattern link B in the solid torus V. Since the complement of a tubular neighbourhood M of the circle U in S^3 is a solid torus standardly embedded in S^3, we can take $V = S^3 - \mathring{M}$, where $\mathring{M}$ is the interior of M. Consequently, we shall represent B in the complement of U in S^3. The pattern link B, in which we are interested, is the two-component framed and oriented link shown in Fig. 4.1; each component of B has preferred framing.

Let ρ_1 and ρ_2 be the irreducible representations which are assigned to the two components of B; then, the analogue of the decomposition (4.9) reads [2]

$$\begin{aligned} W(B;\chi[\rho_1],\chi[\rho_2]) &= \sum_{\rho} F_{\rho_1,\rho_2,\rho}\, W(K;\chi[\rho]) \\ &= W(K;\chi[\rho_1]\,\chi[\rho_2]) \,, \end{aligned} \tag{4.13}$$

where K is the framed core of V which has preferred framing and is oriented as the components of B, see Fig. 4.2. In eq. (4.13) one can easily recognize the struc-

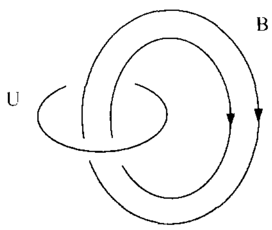

Fig. 4.1. Pattern link B in the complement of U in S^3.

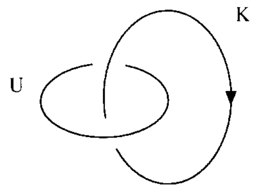

Fig. 4.2. Oriented and framed core K of V.

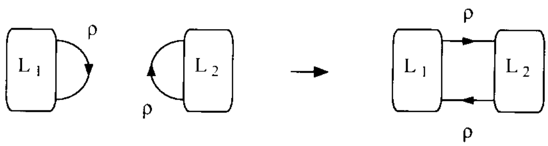

Fig. 4.3. Connected sum of L_1 and L_2.

ture constants (4.3) of $\mathscr{T}$.

For the satellites constructed with the pattern link B, eq. (4.12) takes the form

$$\begin{aligned} E(L') &= E(h^{\diamond}(B), C', ...; \chi[\rho_1], \chi[\rho_2], \chi[\rho'], ...) \\ &= E(C, C', ...; \chi[\rho_1]\chi[\rho_2], \chi[\rho'], ...) \\ &= \sum_{\rho} F_{\rho_1, \rho_2, \rho} E(C, C', ...; \chi[\rho], \chi[\rho'], ...) \,. \end{aligned} \tag{4.14}$$

Since the coupling constant k multiplies the whole action (2.1) (or (2.3)), a modification of the orientation of $\mathbb{R}^3$ is equivalent to a modification of the sign of k; thus, if we denote by $\tilde{L}$ the mirror image of the link L, one has [2]

$$E(\tilde{L})|_k = E(L)|_{-k} \,. \tag{4.15}$$

This equation can easily be verified order by order in perturbation theory.

Finally, let us consider the connected sums of links. Suppose that L_1 and L_2 are two disjoint links and that the oriented components $C \in L_1$ and $C' \in L_2$ are associated with the irreducible representation ρ of the gauge group. Starting from the distant union $L_1 \cup L_2$, we shall now construct a new link which is called the connected sum of L_1 and L_2. Inside some fixed three-ball in $\mathbb{R}^3$, the two components C and C' are cut and glued together as shown in Fig. 4.3. The resulting new link is the connected sum $L_1 \# L_2[\rho]$ of L_1 and L_2 which has been obtained by acting on two link components with colour given by the irreducible representation ρ.

The expectation values for the connected sums of links satisfy the relation [2,4]

$$E(L_1 \# L_2[\rho]) = \frac{E(L_1)E(L_2)}{E_0[\rho]} \,, \tag{4.16}$$

where $E_0[\rho]$ is the expectation value of the Wilson line operator associated with the unknot (circle) in $\mathbb{R}^3$ with standard framing and with colour given by the element $\chi[\rho]$ of the tensor algebra $\mathscr{T}$.

5. Representation ring

The satellite relation (4.14) will play a crucial role in our construction. In fact, we will use systematically the pattern link B, shown in Fig. 4.1, to construct satellites. We shall define a recursive procedure in order to replace each link component, which is associated with a higher dimensional representations of SU(3), with a suitable cabled component. In order to introduce the standard cabling procedure, however, we need to discuss some basic properties of the representation ring of SU(3).

We shall use Dynkin labels to denote the irreducible representations of SU(3). For each couple of nonnegative integers (m, n), the associated irreducible representation corresponds to the highest weight μ given by

$$\mu = m\mu^{(1)} + n\mu^{(2)} , \tag{5.1}$$

where $\mu^{(1)}$ and $\mu^{(2)}$ are the fundamental weights of SU(3). Thus, (0, 0) is the trivial representation, (1, 0) denotes the fundamental representation **3** and (0, 1) its complex conjugate **3***. The representation (m, n) has dimension

$$D(m, n) = \frac{(m+1)(n+1)(m+n+2)}{2} , \tag{5.2}$$

and the value $Q(m, n)$ of the corresponding quadratic Casimir operator is given by

$$Q(m, n) = \frac{m^2+n^2+mn+3(m+n)}{3} . \tag{5.3}$$

Given two representations (m, n) and (a, b), the decomposition of the tensor product $(m, n)\otimes(a, b)$ into a sum of irreducible components can be obtained, for instance, by means of the standard Young tableaux method. The following relations will be useful for our discussion.

For $m \geq 1$ and $n \geq 1$, one has

$$(m, n)\otimes(1, 0) = (m+1, n)\oplus(m-1, n+1)\oplus(m, n-1) , \tag{5.4}$$

$$(m, n)\otimes(0, 1) = (m, n+1)\oplus(m+1, n-1)\oplus(m-1, n) . \tag{5.5}$$

For $m \geq 1$, one obtains

$$(m, 0)\otimes(1, 0) = (m+1, 0)\oplus(m-1, 1) , \tag{5.6}$$

$$(m, 0)\otimes(0, 1) = (m, 1)\oplus(m-1, 0) . \tag{5.7}$$

Finally, for $n \geq 1$, one gets

$$(0, n)\otimes(1, 0) = (1, n)\oplus(0, n-1) , \tag{5.8}$$

$$(0, n)\otimes(0, 1) = (0, n+1)\oplus(1, n-1) . \tag{5.9}$$

Unlike the case of the group SU(2), an explicit formula which gives the decomposition of the tensor product $(m, n)\otimes(a, b)$, for arbitrary representations (m, n) and (a, b) of SU(3), is not known. Thus, for SU(3) one has to analyze, in general, each single tensor product separately. Even if the structure of the SU(3) tensor algebra $\mathcal{T}$ cannot be displayed in compact form, we will show how to derive the relevant properties of $\mathcal{T}$ and of its associated reduced tensor algebra $\mathcal{T}_{(k)}$. In fact, in order to study the properties of the tensor algebra, we only need to consider the relations (5.4)–(5.9).

Let us denote by R the representation ring of SU(3) and $\mathcal{P}(y, \bar{y})$ be the ring of the (finite) polynomials in the two variables y and $\bar{y}$ with integer coefficients. We shall now show that R admits a faithful representation in $\mathcal{P}(y, \bar{y})$. This is a standard result of the theory of simple Lie algebras and its validity is based on the fact that the Lie algebra associated to SU(3) has rank two, of course. Here, we shall just recall the main arguments of the proof in order to illustrate how the recursive use of eqs. (5.4)–(5.9) determines the structure of R and, therefore, the structure of $\mathcal{T}$.

Each irreducible representation of SU(3) is associated to an element of R; the set of the elements which correspond to all the inequivalent irreducible representations of SU(3) is called the standard basis of $\mathcal{T}$. R is a commutative ring with identity; consequently, for each element (m, n) of the standard basis, we only need to give the corresponding representative $[m, n]$ in $\mathcal{P}(y, \bar{y})$. On the one hand, the ring R is generated by the two elements associated with the fundamental weights of SU(3) plus the identity. On the other hand, $\mathcal{P}(y, \bar{y})$ is generated by the two elements y and $\bar{y}$ plus the identity. Thus, the starting point is the obvious correspondence

$$(0,0)\leftrightarrow[0,0]=1\,, \tag{5.10}$$

$$(1,0)\leftrightarrow[1,0]=y\,, \tag{5.11}$$

$$(0,1)\leftrightarrow[0,1]=\bar{y}\,. \tag{5.12}$$

Now, we need to find the representative $[m, n]\in\mathcal{P}(y, \bar{y})$ of a generic representation (m, n). Let us consider firstly the set of representations of the type $(m, 0)$ with $m>1$. From eq. (5.6), one finds

$$[2,0]=[1,0]\cdot[1,0]-[0,1]\,. \tag{5.13}$$

Therefore,

$$[2,0]=y^2-\bar{y}\,. \tag{5.14}$$

In general, eqs. (5.6) and (5.7) imply that

$$[m+2,0]=[m-1,0]+[m+1,0]\cdot[1,0]-[m,0]\cdot[0,1]\,. \tag{5.15}$$

Eq. (5.15) must hold for any $m\geq 1$ and gives a recursive relation for the elements

$\{[m,0]\}$. Since we already know $[2,0]$, $[1,0]$, $[0,1]$ and $[0,0]$, this recursive relation determines $[m,0]$, for arbitrary m, uniquely. For example, one finds

$$[3,0]=y^3-2y\bar{y}+1\ , \tag{5.16}$$

$$[4,0]=y^4-3y^2\bar{y}+\bar{y}^2+2y\ . \tag{5.17}$$

The same argument, based on eqs. (5.8) and (5.9), can be used to find the polynomials $\{[0,\ n]\}$ associated with the representations $\{(0,\ n)\}$, of course. Equivalently, since $(0,\ m)$ is the complex conjugate of $(m,\ 0)$, the polynomial $[0,\ m]$ can be obtained from $[m,\ 0]$ simply by exchanging y with $\bar{y}$ and vice versa. Therefore, at this stage, all the polynomials of the type $\{[m,\ 0]\}$ and $\{[0,\ n]\}$ are uniquely determined for arbitrary m and n.

From eq. (5.7), one obtains

$$[m,1]=[m,0]\cdot[0,1]-[m-1,0]\ . \tag{5.18}$$

This equation, which holds for any $m\geq 1$, permits us to find all the polynomials of the type $\{[m,\ 1]\}$. Similarly, eq. (5.8) defines the recursive relation

$$[1,n]=[0,n]\cdot[1,0]-[0,n-1]\ , \tag{5.19}$$

which uniquely fixes the polynomials $\{[1,n]\}$ for arbitrary n. Finally, let us consider a generic representation $(m,\ n)$; the associated polynomial $[m,\ n]\in\mathscr{P}(y,\ \bar{y})$ can be determined, for instance, by using a recursive procedure in the values of the index n. Indeed, eq. (5.4) gives

$$[m,n+1]=[m,n]\cdot[0,1]-[m+1,n-1]-[m-1,n]\ . \tag{5.20}$$

Assume, by induction, that the polynomials $\{[m,\ n]\}$ are known for arbitrary m and for $n\leq n_0$. Then, eq. (5.20) can be used to find $[m,\ n_0+1]$. On the other hand, $[m,0]$ and $[m,\ 1]$ are known ; therefore, eq. (5.20) permits us to find $[m,\ n]$ for generic values of m and n.

To sum up, the representation ring R of SU(3) can conveniently be described by $\mathscr{P}(y,\ \bar{y})$; we have also proved that the polynomial $[m,\ n]$, which is associated to the generic representation $(m,\ n)$, is uniquely determined by the relations (5.4)–(5.9). A few examples of representative polynomials are in order:

$$[0,2]=\bar{y}^2-y,\qquad [1,1]=y\bar{y}-1\ , \tag{5.21}$$

$$[2,1]=y^2\bar{y}-\bar{y}^2-y\ , \tag{5.22}$$

$$[3,1]=y^3\bar{y}-2y\bar{y}^2-y^2+2\bar{y}\ , \tag{5.23}$$

$$[2,2]=y^2\bar{y}^2-y^3-\bar{y}^3\ . \tag{5.24}$$

In general, one finds

$$[m,n]=\sum_{i+j\leq m+n} a_{ij}y^i\bar{y}^j\ , \tag{5.25}$$

where $\{a_{ij}\}$ are integer numbers and $a_{mn}=1$. The given representation of R in $\mathscr{P}(y, \bar{y})$ is particularly convenient for our purposes. Indeed, each polynomial (5.25) provides the explicit connection between the elements of the standard basis of $\mathscr{T}$ and the elements of the new basis defined in terms of the powers of $(1,0)$ and $(0,1)$, which correspond to the monomials $\{y^i\bar{y}^j\}$. To be more precise, let us denote by $\chi[m, n]$ the element of the standard basis of $\mathscr{T}$ which is associated to the representation (m, n) of SU(3). Eq. (5.25) implies that

$$\chi[m,n]=\sum_{i+j\leq m+n} a_{ij}(\chi[1,0])^i(\chi[0,1])^j\,, \tag{5.26}$$

where the coefficients $\{a_{ij}\}$ appearing in eq. (5.26) coincide with the coefficients $\{a_{ij}\}$ entering eq. (5.25).

6. Computing the link polynomials

The quantum CS field theory is exactly solvable because, with a finite number of operations, one can find the exact expression $E(L)$ for a generic link L. As we have already mentioned, $E(L)$ can easily be computed by using the rules introduced in Refs. [2, 4]; these rules have been derived from the field theory and are in agreement with the results obtained in perturbation theory, of course. In this section, we shall briefly summarize these rules, which are formulated for link diagrams, when the gauge group is $G=\mathrm{SU}(3)$.

It is convenient to represent the ambient isotopy classes of framed links in $\mathbb{R}^3$ by the regular isotopy classes of link diagrams. We shall use the vertical framing convention; this means that the linking number of each link component and its framing is equal to the writhe [2] of the corresponding component of the link diagram. In particular, for the connected sums of links, the vertical framing convention completely specifies the framings of the resulting components of $L_1 \# L_2[\rho]$.

Let us introduce the so-called deformation parameter q which, in the quantum CS field theory, is given by [2]

$$q=\exp(-\mathrm{i}2\pi/k)\,, \tag{6.1}$$

where the renormalized coupling constant k is defined by eqs. (2.8)–(2.10). In order to compute $E(L)$, one can use standard cabling; this method consists of two steps. Firstly, we shall give the rules for the computation of a generic link when each link component has colour specified by the fundamental representation of SU(3). Secondly, we shall introduce a recursive procedure, which is based on the use of the satellite relation (4.14), to compute the link invariants when the colours of the link components are arbitrary.

Theorem 1. *Let L be a link diagram with components $\{C_1, C_2, ..., C_n\}$ in which* *each component is oriented and has colour given by the fundamental representation* **3** *of* SU(3). *The associated expectation value,*

$$E(L)=E(C_1, C_2, ..., C_n; \mathbf{3}, \mathbf{3}, ..., \mathbf{3}) , \tag{6.2}$$

is uniquely determined by
(1) regular isotopy invariance;
(2) covariance under an elementary modification of the writhe;
(3) skein relation;
(4) value of the unknot with zero writhe.

This theorem has been proved in Ref. [4]; let us recall here the precise meaning of points (1)–(4). Regular isotopy invariance means that $E(L)$ is invariant under modifications of the diagrams which are obtained by combining (smooth) isotopy transformations on the plane of the diagram with a finite sequence of Reidemeister moves [2] of types II and III. Consider now a given link diagram L; let us modify the writhe $w(C)$ of a single component C according to $w(C)\to w'(C)=w(C)\pm 1$. Let us denote by $L^{(\pm)}$ the new link diagram which has been obtained from L according to the above procedure. Then, point (2) means that [2]

$$E(L^{(\pm)})=q^{\pm 4/3}E(L) . \tag{6.3}$$

The skein relation mentioned in point (3) is given by [2]

$$q^{1/6}E(L_+)-q^{-1/6}E(L_-)=[q^{1/2}-q^{-1/2}]E(L_0) , \tag{6.4}$$

where the configurations $L_\pm$ and L_0 are shown in Fig. 6.1.

Finally, the normalization of $E(L)$ is fixed by the value $E(U_0; \mathbf{3})$ of the unknot U_0 with writhe equal to zero. This value cannot be chosen arbitrarily but is uniquely fixed [2] by the field theory. For the gauge group SU(3), the value of the unknot is [2]

$$E(U_0; \mathbf{3})=E_0[\mathbf{3}]=q+1+q^{-1} . \tag{6.5}$$

One of the consequences of Theorem 1 is the following:

Property 1. *For any framed, oriented and coloured link L with colours given by the fundamental representation* **3** *of* SU(3), *the associated expectation value E(L) is a finite Laurent polynomial in the variable x given by*

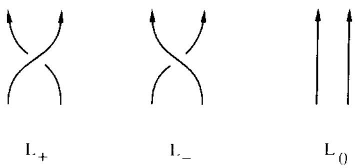

Fig. 6.1. Skein related configurations.

$$x=q^{1/3}\,. \tag{6.6}$$

Proof. The value (6.5) of the unknot with zero writhe is a finite Laurent polynomial in x and, because of eq. (6.3), the value of the unknot with arbitrary writhe also belongs to $Z[x^{\pm 1}]$. By means of the skein relation (6.4) and of eqs. (6.3) and (6.5), one obtains that, for the distant union of an arbitrary number of unknots, the associated invariant belongs to $Z[x^{\pm 1}]$. This result is in perfect agreement with property (4.1), of course. Now, by using the skein relation recursively, $E(L)$ can be written as a finite linear combination of the invariants associated with the distant union of unknots in which each unknot may have a nontrivial writhe. Thus, we only need to prove that the coefficients, entering this linear combination, belong to $Z[x^{\pm 1}]$. Let us consider the standard ascending method [15] to construct $E(L)$. This recursive method is based on the observation that any given link diagram can be transformed into a distant union of unknots provided some overcrossings are exchanged for undercrossings or vice versa. At each step of this recursive procedure, the skein relation (6.4) can be written in the form

$$E(L_+)=q^{-1/3}E(L_-)+(q^{1/3}-q^{-2/3})E(L_0)\,, \tag{6.7}$$

or

$$E(L_-)=q^{1/3}E(L_+)-(q^{2/3}-q^{-1/3})E(L_0)\,. \tag{6.8}$$

Since all the coefficients appearing in eqs. (6.7) and (6.8) belong to $Z[x^{\pm 1}]$, the inductive ascending procedure permits us to express $E(L)$ as a linear combination of the values of the unknots in which all the coefficients entering this linear combination belong to $Z[x^{\pm 1}]$. Therefore, property 1 is proved. □

At this stage, we are able to compute $E(L)$ when the link components have colours which are given by the trivial representation, or the fundamental representation **3** or its complex conjugate **3***. Indeed, each component associated with the trivial representation can be eliminated and each oriented component, associated with **3***, is equivalent to the same component with opposite orientation associated with **3**. Consequently, by means of Theorem 1, we can easily compute $E(L)$.

Let us now consider the case in which the colours of the link components correspond to generic representations $\{(m, n)\}$ of SU(3). In order to compute the expectation value of the associated Wilson line operator, one can use standard cabling. This method is based on the satellite relations and on eq. (5.26). Suppose that a given link component is characterized by an irreducible representation (m, n) of SU(3) which is different from (0, 0), (1, 0) and (0, 1). In this case, this component will be replaced by the image under $h^{\diamond}$ of an appropriate linear combination of pattern links. These pattern links are chosen to have all their components associated with $(1, 0)$, or $(0, 1)$, or $(0, 0)$. Such pattern links always exist and are determined precisely by eq. (5.26). When all the link components which are associated with higher dimensional representations of SU(3) have been substituted, one gets a linear combination of satellites in which all the link components have colours $(1, 0)$, or $(0, 1)$ or $(0, 0)$ and, by means of Theorem 1, one can finally compute the expectation value of the observable. We shall now give the details of this construction.

Let us denote by $B(ij)$ the pattern link shown in Fig. 6.2; $B(ij)$ is defined in the solid torus which coincides with the complement of the circle U in S^3. The link $B(ij)$ has $i+j$ oriented components and each component has preferred framing. The satellite of a knot constructed with the pattern link $B(ij)$ is called a cabled knot. Let us assume that i components of $B(ij)$ have colour $\chi[1, 0]$ and j components have colour $\chi[0, 1]$; we shall denote by $W(B(ij))$ the product of the associated Wilson line operators. According to eq. (4.9), $W(B(ij))$ admits an expansion of the type

$$W(B(ij)) = \sum_{\rho} \xi(\rho) W(K; \chi[\rho]) , \tag{6.9}$$

where K is the core of the solid torus (with preferred framing), shown in Fig. 4.2. The coefficients $\{\xi(\rho)\}$ entering eq. (6.9) can be determined by using eq. (4.13) recursively. So, these coefficients are uniquely determined by the structure constants of the tensor algebra $\mathscr{T}$. Consequently, if the element $\chi[m, n]$ admits the presentation shown in eq. (5.26), one obtains

$$W(K; \chi[m, n]) = \sum_{i+j \leq m+n} a_{ij} W(B(ij); \chi[1, 0], ..., \chi[0, 1], ...) . \tag{6.10}$$

Now, let C be a generic component of a link with colour $\chi[m, n]$. As we have already mentioned, the oriented and framed component C can be understood to be the image $h^{\diamond}(K)$ of the oriented and framed knot K (with colour $\chi[m, n]$) under the homeomorphism $h^{\diamond}$ defined in Section 4. Consequently, eq. (6.10)

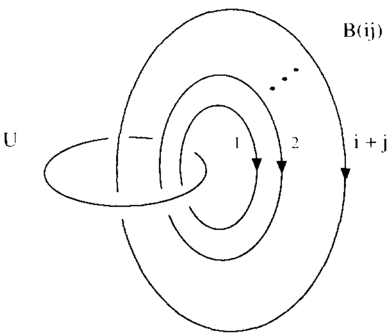

Fig. 6.2. Patterns link $B(ij)$ in the complement of U in S^3.

implies that C can be replaced by a linear combination of cabled components according to

$$C \text{ with colour } [m,n] \leftrightarrow \sum_{i+j \leq m+n} a_{ij} h^{\diamond}(B(ij)) \ . \tag{6.11}$$

Eq. (6.11) gives the desired relation which is satisfied by the CS expectation values of the observables; each link component, which is associated with a higher dimensional representation of SU(3), is equivalent to a certain linear combination of cabled components which have colours given by the fundamental representation **3** or **3***. The coefficients $\{a_{ij}\}$ entering this linear combination are determined by the structure constants of the tensor algebra of the gauge group and are integer numbers.

Theorem 2. *Let L be an oriented link diagram in which an irreducible representation of* SU(3) *is attached to each component. The associated expectation value* $E(L)$ *can be computed by means of standard cabling.* $E(L)$ *represents a regular isotopy invariant of oriented coloured link diagrams; moreover,* $E(L) \in Z[x^{\pm 1}]$.

Proof. According to the standard cabling procedure, each link component which is associated with a higher dimensional representation of SU(3) can be replaced by the combination of cabled components shown in eq. (6.11). Consequently, $E(L)$ can be written as a sum of satellites in which all the components have colours given by the trivial representation, or the representation **3** or its complex conjugate **3***. Theorem 1 then implies that $E(L)$ is a regular isotopy invariant of oriented and coloured link diagrams. Finally, since the coefficients $\{a_{ij}\}$ entering eq. (6.11) are integer numbers, $E(L)$ is a linear combination with integer coefficients of the invariants defined in Theorem 1. Therefore, according to Property 1, $E(L)$ belongs to $Z[x^{\pm 1}]$. □

Theorems 1 and 2 are the reconstruction theorems which determine the expectation values of the observables of the non-Abelian SU(3) CS theory in $\mathbb{R}^3$. Let us consider a few examples of link polynomials. We shall denote by $E_0[m,n]$ the value of the unknot with preferred framing and with colour $\chi[m,n]$. One has

$$E_0[2,0] = E_0[0,2] = (1+q^{-2}) \frac{(1-q^3)}{(1-q)} , \tag{6.12}$$

$$E_0[1,1] = (1+q)^2 (1+q^{-2}) , \tag{6.13}$$

$$E_0[4,1] = E_0[1,4] = \frac{(q^{-5}-1)(1-q^7)}{(1-q)^2} . \tag{6.14}$$

For the right-handed trefoil 3_1 (with vertical framing convention), shown in Fig.

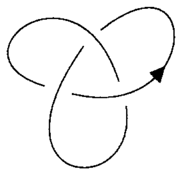

Fig. 6.3. Right-handed trefoil.

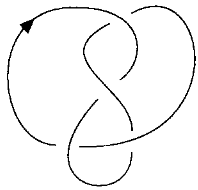

Fig. 6.4. Figure-eight knot.

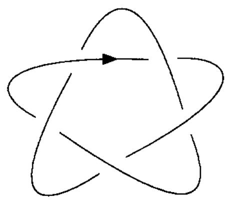

Fig. 6.5. Knot 5_1.

6.3, one finds

$$E(3_1;\chi[1,0]=(q+1+q^{-1})(q^2+1-q^{-2})\ , \tag{6.15}$$

$$E(3_1;\chi[2,0])=\frac{(1+q^{-2})(1-q^3)}{(1-q)} \times[q^6+q^3+q^2-q-q^{-2}-q^{-3}+q^{-5}]\ . \tag{6.16}$$

The figure-eight knot 4_1 is shown in Fig. 6.4; one gets

$$E(4_1;\chi[1,0])=q^4+q^3-1+q^{-3}+q^{-4}\ , \tag{6.17}$$

$$E(4_1;\chi[2,0])=q^{-10}(1+q^{18})\frac{(1-q^3)}{(1-q)} -q^{-7}(1+q^{11})\frac{(1-q^4)}{(1-q)}+q^{-2}(1+q^2)(1+q)^2\ . \tag{6.18}$$

For the knot 5_1, shown in Fig. 6.5, one obtains

$$E(5_1,\chi[1,0])=\frac{(1-q^3)(1+q^2-q^{-4})}{(1-q)}q^{-1/3}\ , \tag{6.19}$$

$$E(5_1;\chi[2,0]) = \frac{(1-q^3)(1+q^2)}{(1-q)}\,q^{-31/3} + \frac{(1-q^3)(1-q^5)}{(1-q)^2}\left[\frac{(1+q^3)}{(1+q)} - q^{-9}\right]q^{8/3}. \qquad (6.20)$$

Finally, for the Whitehead's link 5_1^2, shown in Fig. 6.6, one finds

$$E(5_1^2;\chi[2,0],\chi[1,0]) = \frac{(1+q^2)(1-q^3)}{(1-q)}\,q^{-1/3} + \frac{(1-q^6)}{(1-q)}\,[1-q^2+2q^4-q^6](1+q^2)q^{-19/3}. \qquad (6.21)$$

7. Values of the unknot

In this section we shall compute $E_0[m,n]$, which is the CS expectation value of the Wilson line operator associated with the unknot in $\mathbb{R}^3$ (with preferred framing) with colour $\chi[m,n]$. Our purpose is to produce the expression of $E_0[m,n]$ for arbitrary m and n. In studying the properties of the link polynomials, the values of the unknot play an important role. Indeed, in Section 4 we have seen that, because of general covariance, any coloured link contained inside a solid torus determines a certain colour state associated with the core of the solid torus. Since the unknot is the companion of any link, the values of the unknot represent the basic building blocks of the link polynomials. In order to determine the structure of the reduced tensor algebra $\mathscr{T}_{(k)}$, we shall use the results of this section.

In Section 6 we have given the rules for the computation of a generic coloured link. Thus, for fixed values of m and n, the computation of $E_0[m,n]$ is straightforward. The only nontrivial task is to find the general dependence of $E_0[m,n]$ on the integers m and n. In order to solve this problem, we shall use the symmetry properties of the CS theory and the recursive relations (5.4)–(5.9).

Let us denote by C the oriented unknot in $\mathbb{R}^3$ with preferred framing and colour $\chi[m,n]$. Clearly, the orientation of C can be modified by means of an ambient isotopy transformation. Since a modification of the orientation of C is equivalent

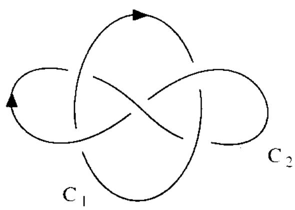

Fig. 6.6. Whitehead's link.

to replace $\chi[m, n]$ with $\chi[n, m]$, one has

$$E_0[m, n] = E_0[n, m] . \tag{7.1}$$

Let us now consider the distant union of two oriented unknots C_1 and C_2 (both with preferred framing) in $\mathbb{R}^3$. Suppose that C_1 has colour $\chi[\rho_1]$ and C_2 has colour $\chi[\rho_2]$. Eq. (4.1) and the satellite relation (4.14) imply that

$$E_0[\rho_1]E_0[\rho_2] = \sum_\rho F_{\rho_1,\rho_2,\rho} E_0[\rho] , \tag{7.2}$$

where $\{F_{\rho_1,\rho_2,\rho}\}$ are the structure constants of the tensor algebra $\mathscr{T}$:

$$\chi[\rho_1]\chi[\rho_2] = \sum_\rho F_{\rho_1,\rho_2,\rho}\chi[\rho] . \tag{7.3}$$

Eq. (7.2) shows that the set $\{E_0[m, n]\}$ of the possible values of the unknot gives a representation in $Z[x^{\pm 1}]$ of the representation ring of the gauge group. Consequently, one can use the relations (5.4)–(5.9) to find the values of the unknot. In fact, the recursive argument that we presented in Section 5 determines the value of $E_0[m, n]$ uniquely. The result is summarized by the following theorem.

Theorem3. *The expectation value $E_0[m, n]$ of the unknot in $\mathbb{R}^3$ with preferred framing and colour $\chi[m, n]$ is given by*

$$E_0[m, n] = q^{-(m+n)} \frac{(1-q^{m+1})(1-q^{n+1})(1-q^{m+n+2})}{(1+q)(1-q)^3} . \tag{7.4}$$

Proof. Clearly, eq. (7.4) gives the correct value of the unknot for the representations (0, 0), (1, 0) and (0, 1). Moreover, one can easily verify [16] that expression (7.4) satisfies the recursive relations (5.4)–(5.9). Consequently, eq. (7.4) represents the unique solution of the recursive relations with the correct initial data. Therefore, eq. (7.4) gives the values of the unknot in the CS theory. □

In agreement with eq. (7.1), the value of $E_0[m, n]$ shown in eq. (7.4) is symmetric in the indices m and n. Since the deformation parameter q is given by eq. (6.1), the expression (7.4) admits a Taylor expansion in powers of $\lambda = (2\pi/k)$ around $\lambda = 0$. Each term of this expansion represents the value of the corresponding Feynman diagrams found in perturbation theory. Let us now verify that the expression (7.4) is really in agreement with the perturbative result (3.5). From eq. (7.4) one gets

$$E_0[m, n] = \frac{(m+1)(n+1)(m+n+2)}{2} \times [1 - \tfrac{1}{12}\lambda^2(m^2+n^2+mn+3m+3n)] + O(\lambda^3) . \tag{7.5}$$

By taking into account eqs. (5.2) and (5.3), one finds that this expression coincides precisely with the perturbative result (3.5).

Eq. (7.4) shows that $E_0[m, n]$ is actually a finite Laurent polynomial in the deformation parameter q. Indeed, the factor $(1+q)(1-q)^3$ in the denominator of the expression (7.4) cancels out the roots at $q=-1$ and $q=1$ of the numerator. The dependence of $E_0[m, n]$ on the CS coupling constant k can be explicitly displayed by writing the expression (7.4) in the equivalent form

$$E_0[m,n]=\frac{1}{2}\frac{\sin[\pi(m+1)/k]\sin[\pi(n+1)/k]\sin[\pi(m+n+2)/k]}{\cos[\pi/k]\sin^3[\pi/k]}\,. \tag{7.6}$$

We conclude this section by proving a property which will be useful to determine the structure of the reduced tensor algebra $\mathcal{T}_{(k)}$.

Property 2. *For any framed, oriented and coloured link L, in which one link component has colour $\chi[m, n]$, the associated expectation value $E(L)$ takes the form*

$$E(L)=E_0[m,n]\mathcal{F}(L)\,, \tag{7.7}$$

where $\mathcal{F}(L)$ is a finite Laurent polynomial in the variable x defined in eq. (6.6).

Proof. Consider the connected sum $L_1 \# L_2$ in which the two links L_1 and L_2 are copies of the link L; in other words, L_1 and L_2 are separately ambient isotopic with the framed link L. Let the connected sum $L_1 \# L_2$ be obtained from L_1 and L_2 by acting on the link component associated with $\chi[m, n]$, Then, from eq. (4.16) it follows that

$$E(L_1 \# L_2)=\frac{E(L)E(L)}{E_0[m,n]}\,. \tag{7.8}$$

Since $E(L)\in Z[x^{\pm 1}]$, one can factorize the maximal power of x^{-1} in $E(L)$ and write

$$E(L)=x^{-a}\mathcal{P}(x)\,, \tag{7.9}$$

where a is a nonnegative integer and $\mathcal{P}(x)$ is an ordinary finite polynomial in x with integer coefficients. Similarly, one has

$$E_0[m,n]=x^{-b}\mathcal{P}_0(x)\,, \tag{7.10}$$

where b is a positive integer and $\mathcal{P}_0$ is a polynomial in x with integer coefficients. As shown in eq. (7.4), one has $b=3(m+n)$; it should be noted that $\mathcal{P}_0(x)$ is not vanishing for $x=0$. By using eqs. (7.9) and (7.10), eq. (7.8) takes the form

$$E(L_1 \# L_2)=x^{-(2a-b)}\frac{\mathcal{P}(x)\mathcal{P}(x)}{\mathcal{P}_0(x)}\,. \tag{7.11}$$

Since $E(L_1 \# L_2) \in Z[x^{\pm 1}]$, eq. (7.11) implies that all the roots of the polynomial $\mathscr{P}_0(x)$ must also be roots of the product $\mathscr{P}(x)\mathscr{P}(x)$. A priori, there are now two possibilities:
(1) $\mathscr{P}(x)/\mathscr{P}_0(x)$ is an ordinary polynomial in x:
(2) $\mathscr{P}(x)/\mathscr{P}_0(x)$ is not a polynomial in x.
In case (1), $\mathscr{P}$ can be divided by $\mathscr{P}_0$ and one has

$$\mathscr{P}(x) = \mathscr{P}_0(x)\,\mathscr{G}(x) , \tag{7.12}$$

where $\mathscr{G}(x)$ is a polynomial in x. Consequently, from eq. (7.9) one obtains

$$E(L) = E_0[m, n]x^{-a+b}\mathscr{G}(x) . \tag{7.13}$$

Therefore, in case (1) one finds that eq. (7.7) is satisfied with $\mathscr{F}(L) = x^{-a+b}\mathscr{G}(x)$.

In order to complete the proof, we shall now show that possibility (2) is never realized; the point is that possibility (2) is not consistent with the connected sum formula (4.16). Indeed, if $\mathscr{P}(x)/\mathscr{P}_0(x)$ is not a polynomial, then it has a pole $(x-x_0)^{-\beta}$ of a certain fixed order β. Since $\mathscr{P}(x)$ multiplied by $\mathscr{P}(x)/\mathscr{P}_0(x)$ is a polynomial, this pole must be cancelled by a root of $\mathscr{P}(x)$. Let us now consider the link L_N which coincides with the connected sum of N copies of the link L, as shown in Fig. 7.1. Eq. (4.16) gives

$$E(L_N) = E(L)\left(\frac{E(L)}{E_0[m, n]}\right)^{N-1} , \tag{7.14}$$

which can be written as

$$E(L_N) = x^{-[Na-(N-1)b]}\mathscr{P}(x)\left(\frac{\mathscr{P}(x)}{\mathscr{P}_0(x)}\right)^{N-1} . \tag{7.15}$$

The same argument that we have used before now implies that $\mathscr{P}(x)$ must eliminate the pole $(x-x_0)^{-(N-1)\beta}$. Since N can be chosen to be arbitrarily large, the cancellation mechanism of the pole cannot take place because $\mathscr{P}(x)$ is a finite polynomial. Thus, possibility (2) is excluded and Proposition 2 is proved. □

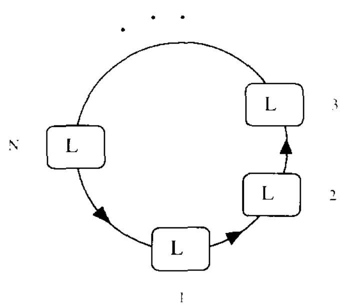

Fig. 7.1. Connected sum of N copies of L.

8. Values of the Hopf link

As we have already mentioned, we shall use Dehn surgery on S^3 to solve the quantum CS theory in a generic three-manifold $\mathcal{M}$. To be more precise, we shall use the operator surgery method [3] to compute the expectation values of the observables in $\mathcal{M}$ by means of the observables in S^3. One of the basic ingredients in the construction of the surgery operators is the so-called Hopf matrix [3]. Let us now recall why the expectation value of the Hopf link plays a crucial role in solving the topological field theory.

Dehn's surgery method essentially consists [17] of removing and sewing solid tori in S^3; thus, we need to consider the properties of the CS expectation values which are related to the different embeddings of solid tori in S^3. Actually, because of the so-called Fundamental Theorem [17], we really need to consider solid tori standardly embedded in S^3. Let us consider a solid torus N standardly embedded in S^3; clearly, its boundary $\partial N \in S^3$ is a two-dimensional torus. Now, the crucial point to be noted is that ∂N is actually the boundary of two solid tori which are both standardly embedded in S^3. The first solid torus is N, of course; the second solid torus is $S^3 - \mathring{N}$, where $\mathring{N}$ is the interior of N.

In conclusion, a solid torus N standardly embedded in S^3 really defines two solid tori: N itself and its complement in S^3. Suppose now that a certain coloured link L_1 is present in N and another coloured link L_2 is contained in $S^3 - \mathring{N}$; we would like to study the properties of the associated expectation value $\langle W(L_1) W(L_2) \rangle$. The symmetry properties of the CS expectation values, that we have mentioned in Section 4, are valid also in the three-sphere S^3. In particular, the generalized satellite relations permit us to find, for each link contained inside a solid torus, the corresponding colour state associated with the core of the solid torus. Consequently, $\langle W(L_1) W(L_2) \rangle$ can be expressed in terms of the values of the Hopf link whose two oriented components represent the cores of the two solid tori N and $S^3 - \mathring{N}$.

To sum up, the values of the Hopf link give us the pairing between the colour states which are associated with two complementary solid tori standardly embedded is S^3. For this reason, the values of the Hopf link are the fundamental ingredients in the construction of the surgery operators and, together with the satellite relations, characterize the topological properties of the quantum CS field theory completely. Thus, our interest is to produce the values of the Hopf link in S^3. Since the CS expectation values in S^3 coincide with the expectation values in $\mathbb{R}^3$ with the constraint that k is an integer, we only need to determine the values of the Hopf link in $\mathbb{R}^3$.

Let us consider the Hopf link in $\mathbb{R}^3$ whose two oriented components $\{C_1, C_2\}$, shown in Fig. 8.1, have preferred framings. Let C_1 have colour $\chi[m, n]$ and C_2 have colour $\chi[a, b]$. We are interested in the associated expectation value

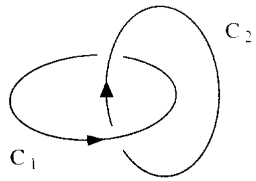

Fig. 8.1. Oriented Hopf link.

$$H[(m,n);(a,b)]=\langle W(C_1;\chi[m,n])W(C_2;\chi[a,b])\rangle|_{\mathbb{R}^3}. \tag{8.1}$$

By means of an ambient isotopy, the components C_1 and C_2 of the Hopf link of Fig. 8.1 can be exchanged; consequently, one has

$$H[(m,n);(a,b)]=H[(a,b);(m,n)]. \tag{8.2}$$

As we have mentioned in Section 4, $E(L)$ is invariant under a global "charge conjugation" transformation which consists of substituting each irreducible representation ρ for its complex conjugate ρ^*. Therefore, one finds

$$H[(m,n);(a,b)]=H[(n,m);(b,a)]. \tag{8.3}$$

For fixed values of (m,n) and (a,b), $H[(m,n);(a,b)]$ can easily be computed by means of the rules specified by the reconstruction theorems. The general dependence of $H[(m,n);(a,b)]$ on the irreducible representations (m,n) and (a,b) is summarized by the following theorem.

Theorem 4. *The expectation value $H[m,n;a,b]$ of the Hopf link in $\mathbb{R}^3$ is given by*

$$\begin{aligned}H[(m,n);(a,b)]&=q^{-[(m+n)(a+b+3)+(m+3)b+(n+3)a]/3}\frac{1}{(1-q)^3(1+q)}\\&\times[1+q^{(n+1)(a+b+2)+(m+1)(b+1)}+q^{(m+1)(a+b+2)+(n+1)(a+1)}\\&-q^{(m+1)(b+1)}-q^{(n+1)(a+1)}-q^{(m+n+2)(a+b+2)}].\end{aligned} \tag{8.4}$$

Proof. In order to prove the validity of eq. (8.4), we shall use eqs. (5.4)–(5.9), which, combined with the satellite relations and the formula (4.16) for the connected sums of links, permit us to find recursive relations for the values $H[(m,n);(a,b)]$. Then, one can verify that the expression (8.4) satisfies these recursive relations and gives the correct values of the initial data. Thus, eq. (8.4) represents the unique solution of the recursive relations which is consistent with the initial values. The algebraic part of the proof is more complicated than the case of the unknot, of course; we shall now give the details of the crucial steps of the proof.

Let us consider the link L shown in Fig. 8.2a. On the one hand, the link L is a satellite of the Hopf link; therefore, by using the satellite formula (4.14), one obtains

$$E(L;\chi[\rho_1],\chi[\rho_2],\chi[\rho_3])=H[\rho_2;\rho_1\otimes\rho_3]\ . \tag{8.5}$$

On the other hand, L is the connected sum of the two Hopf links shown in Fig. 8.2b. Thus according to eq. (4.16), one has (when ρ_2 is irreducible)

$$H[\rho_2;\rho_1\otimes\rho_3]=\frac{H[\rho_1;\rho_2]H[\rho_2;\rho_3]}{E_0[\rho_2]}\ . \tag{8.6}$$

Now, by setting $\rho_1=(m,n)$, $\rho_3=(1,0)$ and $\rho_2=(a,b)$, from eq. (8.6) we have:

$$\frac{H[(m,n);(a,b)]H[(1,0);(a,b)]}{E_0[a,b]}=H[(m+1,n);(a,b)]$$
$$+H[(m-1,n+1);(a,b)]+H[(m,n-1);(a,b)]\ . \tag{8.7}$$

In the same way, if we set $\rho_1=(m-1,n)$, $\rho_3=(0,1)$ and $\rho_2=(a,b)$, we obtain:

$$\frac{H[(m-1,n);(a,b)]H[(0,1);(a,b)]}{E_0[a,b]}=H[(m-1,n);(a,b)]$$
$$+H[(m,n-1);(a,b)]+H[(m-2,n);(a,b)]\ . \tag{8.8}$$

Subtracting eqs. (8.7) and (8.8), we arrive at the promised recursive relation:

$$H[(m+1,n);(a,b)]=H[(m,n);(a,b)]H[(1,0);(a,b)]E_0^{-1}[a,b]$$
$$-H[(m-1,n);(a,b)]H[(0,1);(a,b)]E_0^{-1}[a,b]$$
$$+H[(m-2,n);(a,b)]\ .$$

Similarly, by using eq. (8.3), we find

$$H[(m,n+1);(a,b)]=H[(m,n);(a,b)]H[(0,1);(a,b)]E_0^{-1}[a,b]$$
$$-H[(m,n-1);(a,b)]H[(1,0);(a,b)]E_0^{-1}[a,b]$$
$$+H[(m,n-2);(a,b)]\ . \tag{8.10}$$

By using eqs. (8.9) and (8.10) recursively, one can determine the value of $H[(m,n);(a,b)]$ for $m,n\geq 2$ in terms of the initial values:

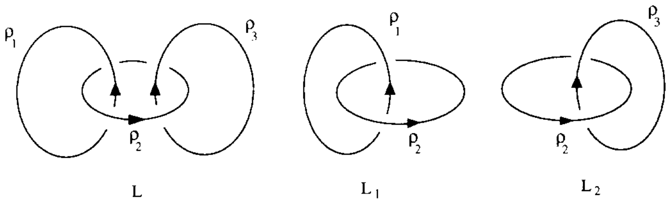

Fig. 8.2. The link L is the connected sum of L_1 and L_2.

$$H[(0,0);(a,b)],\quad H[(1,0);(a,b)],\quad H[(2,0);(a,b)],$$
$$H[(1,1);(a,b)],\quad H[(2,1);(a,b)],\quad H[(2,2);(a,b)]. \tag{8.11}$$

Since any link component with colour $\chi[0,0]$ can be eliminated, we have:

$$H[(0,0);(a,b)]=E_0[a,b]. \tag{8.12}$$

The remaining values of the initial data can easily be calculated by using Theorems 1 and 2 or, simply by using the formula [2]

$$H[\rho_1;\rho_2]=q^{-Q(\rho_1)-Q(\rho_2)}\sum_{\rho\in\rho_1\otimes\rho_2} q^{Q(\rho)}E_0[\rho]. \tag{8.13}$$

The result is [16]:

$$H[(1,0);(a,b)]=q^{-(1+5b/3+4a/3)} \times\frac{(1-q^{1+b})(1-q^{1+a})(1-q^{2+a+b})}{(1-q)^3(1+q)}(1+q^{1+b}+q^{2+a+b}), \tag{8.14}$$

$$H[(2,0);(a,b)]=\frac{(1+q^{1+b}+q^{2+2b}+q^{2+a+b}+q^{2(2+a+b)}+q^{3+2b+a})}{(1-q)^3(1+q)} \times q^{-(2+7b/3+5q/3)}(1-q^{1+b})(1-q^{1+a})(1-q^{2+a+b}), \tag{8.15}$$

$$H[(1,1);(a,b)] =\frac{(1-q^{1+b})(1-q^{1+a})(1-q^{2+a+b})(1+q^{1+b})(1+q^{1+a})}{(1-q)^3(1+q)} \times(1+q^{2+a+b})q^{-2(1+a+b)}, \tag{8.16}$$

$$H[(2,1);(a,b)]=\frac{q^{-(3+8b/3+7a/3)}(1-q^{1+m})(1-q^{1+n})(1-q^{2+a+b})}{(1-q)^3(1+q)} \times(1+q^{1+b}+q^{2+2b}+q^{1+a}+2q^{2+b+a}+2q^{3+2b+a}+q^{4+3b+a} +q^{3+b+2a}+2q^{4+2b+2a}+q^{5+3b+2a}+q^{5+2b+3a}+q^{6+3b+3a}), \tag{8.17}$$

$$H[(2,2);(a,b)]=\frac{q^{-(4+3b+3a)}(1-q^{1+b})(1-q^{1+a})(1-q^{2+a+b})}{(1-q)^3(1+q)} \times(1+q^{1+b}+q^{2+2b}+q^{1+a}+2q^{2+a+b}+2q^{3+2b+a}+q^{4+3b+a}+q^{2+2a} +3q^{4+2b+2a}+2q^{5+3b+2a}+q^{6+4b+2a}+q^{4+b+3a}+2q^{5+2b+3a}+2q^{6+3b+3a} +q^{7+4b+3a}+q^{6+2b+4a}+q^{7+3b+4a}+q^{8+4b+4a}+2q^{3+b+2a}). \tag{8.18}$$

Eqs. (8.9) and (8.10) together with the initial data (8.11) determine the values of the Hopf link uniquely. Now, one can verify [16] that expression (8.4) satisfies the recursive relations (8.9) and (8.10). Moreover, for (m, n) equal to (0, 0), (1.0), (2, 0), (1, 1), (2, 1), and (2, 2), eq. (8.4) reproduces the correct initial values. Consequently, eq. (8.4) represents the values of the Hopf link in the CS theory. □

The value of the Hopf link shown in eq. (8.4) satisfies the symmetry properties (8.2) and (8.3). As usual, expression (8.4) admits a Taylor expansion in powers of $\lambda = (2\pi/k)$ around $\lambda = 0$. We have verified that the first three terms of this expansion agree with the perturbative result (3.7).

We conclude this section by recalling that several properties of the link polynomials defined by Theorems 1 and 2 have been discussed in Ref. [4]. In particular, the computation of $E(L)$ can be simplified by means of the rules introduced in [2]; for example, instead of using standard cabling, $H[\rho_1; \rho_2]$ can be determined by means of eq. (8.13). It should be noted that the covariance property of $E(L)$ under an elementary modification of the writhe can be expressed in the following general form. Let L be an oriented link diagram in which the component C has writhe $w(C)$ and colour $\chi[m, n]$. Consider now the new link diagram $L^{(\pm)}$ which has been obtained from L by means of an elementary modification of the writhe of C:

$$w(C) \to w'(C) = w(C) \pm 1 . \tag{8.19}$$

One has [2]

$$E(L^{(\pm)}) = q^{\pm Q(m,n)} E(L) , \tag{8.20}$$

where $Q(m, n)$ is the value of the quadratic Casimir operator in the irreducible representation (m, n) of SU(3).

9. Chern–Simons theory in the three-sphere

The three-sphere S^3 is a homogeneous space and its fundamental group is trivial. Any link in S^3 is contained inside a three-ball and any link in $\mathbb{R}^3$ is also contained inside a three-ball. The topological properties of links in S^3 and of links in $\mathbb{R}^3$ are equivalent.

The short-distance behaviour of the quantum CS field theory in S^3 is equal to the behaviour in $\mathbb{R}^3$. Therefore, the CS theory in S^3 is renormalizable and finite. Since the model is defined by the same Lagrangian density (2.1) as in $\mathbb{R}^3$, the physical consequences of the renormalized Schwinger–Dyson equations in S^3 coincide with those in $\mathbb{R}^3$. To be more precise, the exterior derivative of the expression corresponding to the propagator of the vector fields must represent an intersection form of a one-dimensional manifold (a given component of a link) and a two-dimensional manifold (the Seifert surface associated with a different link component). This structure of the propagator, which can easily be verified in $\mathbb{R}^3$ by direct inspection (see eq. (2.11)), is a consequence of the Schwinger–Dyson equations. Since S^3 is a homology sphere, the vector field propagator in S^3 must have precisely the same structure as the propagator in $\mathbb{R}^3$.

All the general properties of the expectation values of the observables in $\mathbb{R}^3$ are also valid in S^3. Consequently, Theorems 1 and 2 determine the Wilson line expectation values in $\mathbb{R}^3$ and in S^3 as well. More generally, the rules introduced in [2,4] to compute the link polynomials can also be used in S^3. It is important to note that the numerical input of these rules exclusively depends on the complex parameter q. A change in the value of the coupling constant k does not modify the structure of the link polynomials; it only modifies the numerical values of the deformation parameter q. Now, because of gauge invariance, the coupling constant k of the CS theory in S^3 must take [1] integer values. Thus, the global structure of S^3 does not modify any of the results obtained in $\mathbb{R}^3$; it simply introduces the constraint that k must be an integer.

To sum up, the exact solution of the quantum CS theory in S^3 can be obtained from the solution in $\mathbb{R}^3$ by imposing the condition that k is an integer. We shall fix the orientation of S^3 by adopting the usual right-handed rule to compute linking numbers. Clearly, a modification of the orientation of S^3 is equivalent to replace k with $-k$ in the action (2.1). Moreover, the value $k=0$ must be excluded because, when $k=0$, the action (2.1) vanishes and thus a meaningful CS theory does not exist. Consequently, we may assume that k is positive. Therefore, the relevant values of k that we need to consider are

$$k=\text{integer}, \qquad k=\text{positive}. \tag{9.1}$$

Condition (9.1) does not introduce any singularities in the expectation values of the observables because $E(L)$ is well defined for any nontrivial value of the coupling constant k. Thus, for fixed integer k, the set of observables in S^3 consists of an infinite set of complex numbers; these numbers coincide with the values of $E(L)$ for all the possible links $\{L\}$ in S^3 and for arbitrary irreducible representations of SU(3) attached to the components of these links.

The labels, which are used to distinguish the inequivalent irreducible representations of the gauge group, can be understood as gauge-invariant quantum numbers which can be assigned to the link components. In fact, as a linear space, the tensor algebra $\mathscr{T}$ can be interpreted as a (gauge-invariant) state space. This space is infinite dimensional since there is an infinite number of inequivalent irreducible representations of SU(3). Eq. (4.7) simply means that any gauge invariant state can be written as a linear combination of the states of the standard basis of $\mathscr{T}$. When the deformation parameter q is a free indeterminate, different elements of the standard basis of $\mathscr{T}$ represent physically inequivalent (gauge invariant) states associated with a knot (or a solid torus). For fixed integer k, however, two different elements of $\mathscr{T}$ not necessarily correspond to different values of the observables. For example, let us consider the elements $\chi[0,1]$ and $\chi[4,1]$ of $\mathcal{T}$. Let C be one component of a generic link L in S^3; all the remaining link components are associated with certain fixed representations of the gauge group. When C is associated with $\chi[0,1]$, we shall denote the resulting coloured link by L_1. On the other hand, when the component C is associated with $\chi[4,1]$ we shall denote the resulting coloured link by L_2. For fixed $k=4$, we shall show that $E(L_1)=E(L_2)$ for arbitrary link L. Thus, when $k=4$, $\chi[0,1]$ and $\chi[4,1]$ are physically equivalent because there is no gauge invariant observable which can distinguish $\chi[0,1]$ from $\chi[4,1]$.

For fixed integer k, not all the elements of the standard basis of $\mathcal{T}$ correspond to physically inequivalent states. In order to determine the relevant quantum numbers associated with a knot when k is a fixed integer, we shall introduce an equivalence relation between the elements of the reduced tensor algebra. Two elements χ and χ' will be called physically equivalent if the following equation

$$\langle W(C;\chi)W(C_1;\chi_1)\cdots W(C_m;\chi_m)\rangle|_{S^3}$$
$$=\langle W(C;\chi')W(C_1;\chi_1)\cdots W(C_m;\chi_m)\rangle|_{S^3} \tag{9.2}$$

holds (with fixed k) for any link L with components $\{C, C_1, ..., C_m\}$ and for arbitrary $\{\chi_1, ..., \chi_m\}$. Physically equivalent elements will be denoted by $\chi\sim\chi'$. Note that our definition of physical equivalence has a real physical meaning. Indeed, all the properties of any gauge theory are determined by the set of the gauge invariant observables exclusively. Thus if $\chi\sim\chi'$, eq. (9.2) shows that there is no experiment which can distinguish χ from χ'. For fixed integer k, we shall decompose $\mathcal{T}$ into classes of physically equivalent elements. The resulting set of these classes has an algebra structure inherited from $\mathcal{T}$ and is called the reduced tensor algebra $\mathcal{T}_{(k)}$. Our main purpose now is to determine the structure of the reduced tensor algebra $\mathcal{T}_{(k)}$ associated with SU(3) for each value (9.1) of the coupling constant k.

10. Zeroes of the unknot

For fixed integer k, the value of the unknot $E_0[m,n]$ may vanish for certain values of m and n. In this section, we shall consider the set $\mathcal{A}$ of elements $\{\chi[m,n]\}$ for which the associated value $E_0[m,n]$ is vanishing. In order to construct the reduced tensor algebra $\mathcal{T}_{(k)}$, the knowledge of $\mathcal{A}$ is important. Indeed, because of Property 2, each element of $\mathcal{A}$ is a representative of the null class of physically equivalent states.

Let us first consider the case in which $k\geq 3$. From eq. (7.6) it follows that $E_0[m,n]$ vanishes when one of the following possibilities is satisfied
(1) $m=ka-1$,

(2) $n=kb-1$,
(3) $m+n=kc-2$,
where a, b and c are integers. Let us represent the irreducible representation (m, n) by the points of a two-dimensional square lattice. The elements of $\mathscr{A}$ determine a regular structure on this lattice. The zeroes described in point (1) lie on vertical lines which are equally spaced; those described in point (2) lie on (equally spaced) horizontal lines and the zeroes in point (3) correspond to diagonal lines. The pattern of the zeroes is shown in Fig. 10.1.
The region of the plane delimited by the two axes $m=0$, $n=0$ and the diagonal line $m+n=k-2$ does not contain zeros of the unknot and will be denoted by Δ_k.

The region Δ_k is called the fundamental domain; we will show that the classes represented by the points in Δ_k form a complete basis of $\mathscr{T}_{(k)}$.

When $k=2$, eq. (7.6) gives

$$\lim_{k\to 2} E_0[m,n]=\begin{cases} 0 & \text{for } m \text{ and } n \text{ odd;} \\ -(n+1)/2 & \text{for } n \text{ odd and } m \text{ even;} \\ -(m+1)/2 & \text{for } m \text{ odd and } n \text{ even;} \\ -(m+n+2)/2 & \text{for } m \text{ and } n \text{ even.} \end{cases} \tag{10.1}$$

Finally, for $k=1$ one has

$$\lim_{k\to 1} E_0[m,n]=\tfrac{1}{2}(m+1)(n+1)(m+n+2)=D(m,n)\ . \tag{10.2}$$

It should be noted that the cases in which $k=2$ and $k=1$ present certain peculiarities with respect to the general situation that one has for $k\geq 3$. Consequently, in the construction of the reduced tensor algebra we will need to distinguish the cases $k=1$, $k=2$ and $k\geq 3$.

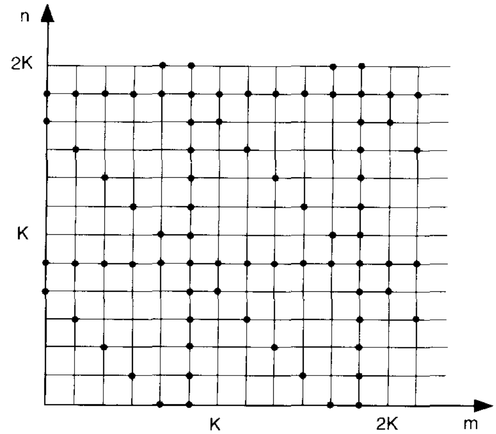

Fig. 10.1. Zeroes of the unknot in a two-dimensional square lattice.

11. Null vectors

In this section we will prove that, when $k \geq 3$, all the elements of $\mathscr{A}$ are generated [16] by two fundamental nontrivial "null vectors".

Property 3. *For $k \geq 3$, each element $\chi[m, n]$ of $\mathscr{A}$ can be written in the form*

$$\chi[m, n] = \zeta_1 \chi_1 + \zeta_2 \chi_2 , \tag{11.1}$$

where

$$\zeta_1 = \chi[k-2, 0] , \tag{11.2}$$

$$\zeta_2 = \chi[k-1, 0] , \tag{11.3}$$

and the elements χ_1 and χ_2 belong to the representation ring R of SU(3).

Proof. All the elements of R which can be written as in eq. (11.1) form a ideal of R denoted by $\mathscr{B}(\zeta_1, \zeta_2)$. We need to prove that each element of $\mathscr{A}$ belongs to $\mathscr{B}(\zeta_1, \zeta_2)$. In order to do this, we shall use the tensor product decompositions

$$(m, n) \otimes (1, 0) = (m+1, n) \oplus (m-1, n+1) \oplus (m, n-1) , \tag{11.4}$$

$$(m, n) \otimes (0, 1) = (m, n+1) \oplus (m+1, n-1) \oplus (m-1, n) . \tag{11.5}$$

Let us recall that the elements of $\mathscr{A}$ correspond to the points on diagonal, horizontal and vertical lines of the square lattice introduced in Section 10. First, of all, we use eqs. (11.4) and (11.5) to prove that if three elements of $\mathscr{B}(\zeta_1, \zeta_2)$ are represented by three consecutive points in a diagonal line, then all the remaining points of the diagonal belong to $\mathscr{B}(\zeta_1, \zeta_2)$. Indeed, setting $m \to m+1$, $n \to n-1$ in eqs. (11.4), (11.5) and subtracting them, we have

$$\begin{aligned} &\chi[m, n]\chi[1, 0] - \chi[m+1, n-1]\chi[0, 1] \\ &\quad = \chi[m-1, n+1] - \chi[m+2, n-2] . \end{aligned} \tag{11.6}$$

Thus, if $\chi[m-1, n+1]$, $\chi[m, n]$ and $\chi[m+1, n-1]$ belong to $\mathscr{B}(\zeta_1, \zeta_2)$, eq. (11.6) implies that $\chi[m+2, n-2]$ also belongs to $\mathscr{B}(\zeta_1, \zeta_2)$ (see Fig. 11.1). With the substitution $m \to m-1$ and $n \to n-1$ in eq. (11.6), we conclude that $\chi[m-2, n+2]$ also belongs to $\mathscr{B}(\zeta_1, \zeta_2)$ as shown in Fig. 11.2. Clearly, al the remaining points on the diagonal can be obtained by induction. The same argument can be used to prove that the presence of three consecutive elements of $\mathscr{B}(\zeta_1, \zeta_2)$ in a

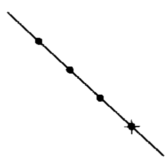

Fig. 11.1. Graphical representation of the recursive relation (11.6).

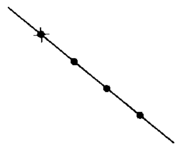

Fig. 11.2. Elements of $\mathscr{B}(\xi_1, \zeta_2)$ on a diagonal line.

vertical (or horizontal line) implies that all the remaining points of the line are also elements of $\mathscr{B}(\zeta_1, \zeta_2)$. For a vertical line the relevant formula is:

$$\chi[m, n]\chi[1, 0]-\chi[m, n+1]\chi[0, 1]=\chi[m, n-1]-\chi[m, n+2] , \qquad (11.7)$$

whereas for a horizontal line the recursive formula is

$$\chi[m, n]\chi[1, 0]-\chi[m-1, n]\chi[0, 1]=\chi[m+1, n]-\chi[m-2, n] . \qquad (11.8)$$

Now we are ready to express all the elements of $\mathscr{A}$ in terms of ζ_1 and ζ_2. Let us consider the elements of the "first" diagonal $m+n=k-2$ (see Fig. 10.1); the fundamental null vector $\zeta_1=\chi[k-2, 0]$ corresponds to the crossing point of the diagonal with the m-axis. From the equation

$$\zeta_1\chi[1, 0]=\zeta_2+\chi[k-3, 1] \qquad (11.9)$$

it follows that $\chi[k-3, 1]$, which belongs to $\mathscr{A}$, also belongs to $\mathscr{B}(\zeta_1, \zeta_2)$. The equation

$$\chi[k-3, 1]\chi[1, 0]-\zeta_1\chi[0, 1]=\chi[k-4, 2] \qquad (11.10)$$

implies that $\chi[k-4, 2]\in\mathscr{B}(\zeta_1, \zeta_2)$. At this stage, we have shown that three elements of $\mathscr{A}$, which are represented by three consecutive points on the first diagonal line, belong to $\mathscr{B}(\zeta_1, \zeta_2)$. Consequently, all the points of this diagonal line belong to $\mathscr{B}(\zeta_1, \zeta_2)$. In other words, the elements of $\mathscr{A}$ corresponding to the points of the first diagonal line (see Fig. 10.1) are also elements of $\mathscr{B}(\zeta_1, \zeta_2)$.

Let us consider now the points lying on the first vertical line $m=k-1$. The first base point is $\zeta_2=\chi[k-1, 0]$. The second point can be obtained by using:

$$\zeta_2\chi[0, 1]=\chi[k-1, 1]+\zeta_1 \quad \Rightarrow \quad \chi[k-1, 1]\in\mathscr{B}(\zeta_1, \zeta_2) . \qquad (11.11)$$

The third point can be expressed in terms of ζ_1 and ζ_2 according to

$$\chi[k-1, 1]\chi[0, 1]=\chi[k-1, 2]+\chi[k, 0]+\chi[k-2, 1] ,$$

$$\zeta_2\chi[1, 0]=\chi[k, 0]+\chi[k-2, 1]$$

$$\Rightarrow \quad \chi[k-1, 2]=\chi[k-1, 1]\chi[0, 1]-\zeta_2\chi[1, 0]\in\mathscr{B}(\zeta_1, \zeta_2) . \qquad (11.12)$$

Since $\chi[k-1, 0]$, $\chi[k-1, 1]$ and $\chi[k-1, 2]$ belong to $\mathscr{B}(\zeta_1, \zeta_2)$, all the points of the vertical line $(k-1, n)$ belong to $\mathscr{B}(\zeta_1, \zeta_2)$.

We now examine the first horizontal line $n=k-1$; for the first point we have

$$\chi[0, k-2]\chi[0, 1]=\chi[0, k-1]+\chi[1, k-3]$$

$$\Rightarrow \quad \chi[0,k-1] \in \mathscr{B}(\zeta_1,\zeta_2)\,. \tag{11.13}$$

The second point can be expressed as

$$\chi[0,k-1]\chi[1,0] = \chi[1,k-1] + \chi[0,k-2]$$
$$\Rightarrow \quad \chi[1,k-1] \in \mathscr{B}(\zeta_1,\zeta_2)\,. \tag{11.14}$$

For the third point we get

$$\chi[1,k-1]\chi[1,0] = \chi[1,k-2] + \chi[0,k-2] + \chi[2,k-1]\,,$$
$$\chi[0,k-1]\chi[0,1] = \chi[1,k-2] + \chi[0,k-2] \quad \Rightarrow \quad \chi[2,k-1]$$
$$= \chi[1,k-1]\chi[1,0] - \chi[0,k-1]\chi[0,1] \in \mathscr{B}(\zeta_1,\zeta_2)\,. \tag{11.15}$$

At this stage, by using eq. (11.8) recursively, we find that all the points of the first horizontal line belong to $\mathscr{B}(\zeta_1,\zeta_2)$.

To sum up, in this first step we have shown that the elements of $\mathscr{A}$ which correspond to the points of the first diagonal, vertical and horizontal lines belong to $\mathscr{B}(\zeta_1,\zeta_2)$.

In the second step, we consider all the remaining diagonal and horizontal lines. Let us start with the base points of the second diagonal line. The element given by $\chi[k-1,k-1]$ lies on the first horizontal line, therefore it belongs to $\mathscr{B}(\zeta_1,\zeta_2)$. The second base point corresponds to $\chi[k,k-2]$; indeed,

$$\chi[k-1,k-2]\chi[1,0] = \chi[k,k-2] + \chi[k-2,k-1] + \chi[k-1,k-3]$$
$$\Rightarrow \quad \chi[k,k-2] = \chi[k-1,k-2]\chi[1,0] - \chi[k-2,k-1]$$
$$-\chi[k-1,k-3] \in \mathscr{B}(\zeta_1,\zeta_2)\,. \tag{11.16}$$

For the last element, we have

$$\chi[k-2,k-1]\chi[0,1] = \chi[k-2,k] + \chi[k-1,k-2] + \chi[k-3,k-1]$$
$$\Rightarrow \quad \chi[k-2,k] \in \mathscr{B}(\zeta_1,\zeta_2)\,. \tag{11.17}$$

By using eq. (11.6) recursively, one can complete the second diagonal line. The same argument that we have presented before can be used to analyze all diagonal and horizontal lines. Indeed, since all the points of the second diagonal belong to $\mathscr{B}(\zeta_1,\zeta_2)$, we can construct the second horizontal line as we did for the first. Then, we can construct the third diagonal line and so on. It is clear that this recursive procedure shows that all the points on the diagonal and horizontal lines belong to $\mathscr{B}(\zeta_1,\zeta_2)$.

In the third and final step, we consider the vertical lines. Three base points for the second vertical line can be obtained by exploring the crossing between the third diagonal and the first horizontal lines. The first element is:

$$\chi[2k-1,k-1]\chi[1,0]=\chi[2k,k-1]+\chi[2k-2,k]+\chi[2k-1,k-2]$$
$$\Rightarrow \quad \chi[2k-1,k-2]\in\mathscr{B}(\zeta_1,\zeta_2) . \tag{11.18}$$

For the second element, we take

$$\chi[2k-1,k-2]\chi[1,0]=\chi[2k,k-2]+\chi[2k-2,k-1]+\chi[2k-1,k-3]$$
$$\Rightarrow \quad \chi[2k-1,k-2]\in\mathscr{B}(\zeta_1,\zeta_2) . \tag{11.19}$$

The third element, which corresponds to $\chi[2k-1, k-1]$, belongs to the first horizontal line. Therefore, by using eq. (11.7), we can complete the second vertical line. Repeating the same argument at every crossing point between the first horizontal line and the diagonal lines, all the vertical lines can be constructed.

In conclusion, all, the elements of $\mathscr{A}$ belong to $\mathscr{B}(\zeta_1, \zeta_2)$ or, in other words, can be written in the form shown in eq. (11.1). Consequently, Property 3 is proved. □

12. Reduced tensor algebra for $k \geq 3$

In this section we will determine the structure of the reduced tensor algebra $\mathscr{T}_{(k)}$ for $k\geq 3$. Let L be a coloured and framed link in S^3 with components $\{C, C', ...\}$ in which the component C has colour $\eta\in\mathscr{T}$. If (for fixed integer k) $\eta=\chi[m, n]\in\mathscr{A}$, Property 2 implies that $\langle W(L)\rangle|_{S^3}=0$ for any link L. As we have already mentioned, this means that $\eta=\chi[m, n]\in\mathscr{A}$ is physically equivalent to the null vector.

Property 4. *If the component C of a link L has colour $\eta=\chi_1\chi_2$ with $\chi_1\in\mathscr{A}$ and $\chi_2\in\mathscr{T}$, one has*

$$\langle W(L)\rangle|_{S^3}=0 . \tag{12.1}$$

Proof. By using the satellite formula (4.14), the expectation value of $W(L)$ can be expressed as

$$\langle W(L)|_{S^3}\equiv\langle W(C, C', ...; \chi_1\chi_2, \chi'...)\rangle|_{S^3}$$
$$=\langle W(h^{\diamond}(B), C', ...; \chi_1, \chi_2, \chi'...)\rangle|_{S^3}\equiv\langle W(L')\rangle|_{S^3} , \tag{12.2}$$

where $h^{\diamond}$ is the homeomorphism defined in Section 4 and B is the two-component pattern link shown in Fig. 4.1. In eq. (12.2), L' denotes the satellite which has been obtained from L by replacing the component C with the image $h^{\diamond}(B)$ of the pattern link. Note that L' has two components which have colours χ_1 and χ_2. Therefore, if $\chi_1\in\mathscr{A}$, by using Property 2 one has $\langle W(L')\rangle|_{S^3}=0$ and, consequently, $\langle W(L)\rangle|_{S^3}=0$. □

In general one can introduce, for fixed integer k, the set $I_{(k)}$ of elements of $\mathscr{T}$ which are characterized by the following property: $\zeta \in I_{(k)}$ if the equation

$$\langle W(C;\zeta) W(C_1;\chi_1) \cdots W(C_m;\chi_m) \rangle |_{S^3} = 0 \tag{12.3}$$

holds for any link L in S^3 with components $\{C, C_1, ..., C_m\}$ and for arbitrary colour states $\{\chi_1, ..., \chi_m\} \subset \mathscr{T}$. In other words, $I_{(k)}$ is the set of elements of $\mathscr{T}$ which are physically equivalent to zero. By using the satellite relations, it is easy to verify that $I_{(k)}$ is an invariant subalgebra (ideal) of $\mathscr{T}$. By definition, the elements $\{\Psi\}$ of the reduced tensor algebra $\mathscr{T}_{(k)}$ are the classes of physically equivalent vectors; these classes coincide with the elements of $\mathscr{T}$ modulo the ideal $I_{(k)}$. Thus, the problem of finding $\mathscr{T}_{(k)}$ is equivalent to the determination of $I_{(k)}$.

Clearly, Property 4 implies that any element η of the form $\eta = \chi_1 \chi_2$, with $\chi_1 \in \mathscr{A}$ and $\chi_2 \in \mathscr{T}$, belongs to $I_{(k)}$. Since any element of $\mathscr{A}$ is of the form (11.1), it is natural to expect that each element ζ of $I_{(k)}$ can be expressed in the form

$$\zeta = \zeta_1 \chi + \zeta_2 \chi' , \tag{12.4}$$

where χ and χ' belong to $\mathscr{T}$. We will prove that this is indeed the case; in other words, for fixed integer $k \geq 3$, $I_{(k)}$ is the ideal generated by the two null vectors ζ_1 and ζ_2 shown in eqs. (11.2) and (11.3). The proof essentially consists of two steps. Firstly, assuming that each element of $I_{(k)}$ is of the form (12.4), we shall determine the corresponding set $\mathscr{T}_{(k)}$ of equivalence classes. Secondly, we will show that this set is physically irreducible; i.e. $\Psi \sim 0$ implies $\Psi \equiv 0$.

We shall now give three basic rules which connect, for fixed integer $k \geq 3$, physically equivalent elements of $\mathscr{T}$. Let us recall that each element $\chi[m, n]$ of the standard basis of $\mathscr{T}$ is represented by a point in the square lattice shown in Fig. 10.1. Furthermore, the elements of $\mathscr{A}$ are organized in diagonal, vertical and horizontal lines in the same lattice. To each line is associated [16] a correspondence rule.

Rule 1. *Let us consider the element $\chi[m, n]$. Suppose that, for some nonnegative integer p, $\chi[m-p, n]$ belongs to $\mathscr{A}$ and is represented by a point on a diagonal line; then*

$$\chi[m, n] \sim -\chi[m-p, n-p] . \tag{12.5}$$

Proof. When $p = 1$, one has

$$\begin{aligned} 0 &\sim \chi[m-1, n]\chi[1, 0] \\ &= \chi[m, n] + \chi[m-2, n+1] + \chi[m-1, n-1] . \end{aligned} \tag{12.6}$$

Since $\chi[m-2, n+1]$ is represented by a point on the diagonal which is determined by $\chi[m-1, n]$, $\chi[m-2, n+1]$ belongs to $\mathscr{A}$ and then $\chi[m-2, n+1] \sim 0$. Therefore, eq. (12.6) gives

$$\chi[m,n] \sim -\chi[m-1,n-1] , \tag{12.7}$$

which shows that eq. (12.5) is satisfied for $p=1$. We now proceed by induction. Let us assume that eq. (12.5) is valid for $p \leq \bar{p}$. We consider the following decompositions

$$\chi[m,n]\chi[1,0] = \chi[m+1,n] + \chi[m-1,n+1] + \chi[m,n-1] , \tag{12.8}$$

$$\chi[m-\bar{p},n-\bar{p}]\chi[1,0] = \chi[m-\bar{p}+1,n-\bar{p}] + \chi[m-\bar{p}-1,n-\bar{p}+1] + \chi[m-\bar{p},n-\bar{p}-1] . \tag{12.9}$$

By induction hypothesis, one has

$$\begin{aligned} \chi[m,n] &\sim -\chi[m-\bar{p},n-\bar{p}] , \\ \chi[m-1,n+1] &\sim -\chi[m-1-\bar{p},n+1-\bar{p}] , \\ \chi m,n-1] &\sim -\chi[m-(\bar{p}-1),n-1-(\bar{p}-1)] . \end{aligned} \tag{12.10}$$

Therefore, by adding eqs. (12.8) and (12.9) and by using (12.10), one finds

$$\chi[m+1,n] \sim -\chi[m+1-(\bar{p}+1),n-(\bar{p}+1)] . \tag{12.11}$$

Eq. (12.11) shows that eq. (12.5) holds also for $p=\bar{p}+1$; this concludes the proof. □

Rule 2. *Let us consider the element* $\chi[m,n]$. *Suppose that, for some nonnegative integer* p, $\chi[m-p,n]$ *belongs to* $\mathscr{A}$ *and is represented by a point on a vertical line; then*

$$\chi[m,n] \sim -\chi[m-2p,n+p] . \tag{12.12}$$

Proof. When $p=1$, one has

$$\begin{aligned} 0 &\sim \chi[m-1,n]\chi[1,0] \\ &= \chi[m,n] + \chi[m-2,n+1] + \chi[m-1,n-1] . \end{aligned} \tag{12.13}$$

Since $\chi[m-1,n-1]$ is represented by a point on the vertical line which is determined by $\chi[m-1,n]$, $\chi[m-1,n-1]$ belongs to $\mathscr{A}$ and then $\chi[m-1,n-1] \sim 0$. Therefore, eq. (12.13) gives

$$\chi[m,n] \sim -\chi[m-2,n+1] , \tag{12.14}$$

which shows that eq. (12.12) is satisfied for $p=1$. We now proceed by induction. Let us assume that eq. (12.13) is valid for $p \leq \bar{p}$. We consider the following decompositions

$$\chi[m,n]\chi[1,0] = \chi[m+1,n] + \chi[m-1,n+1] + \chi[m,n-1] , \tag{12.15}$$

$$\chi[m-2\bar{p}, n+\bar{p}]\chi[1,0]=\chi[m-2\bar{p}+1, n+\bar{p}] +\chi[m-2\bar{p}-1, n+\bar{p}+1]+\chi[m-2\bar{p}, n+\bar{p}-1] . \tag{12.16}$$

By the induction hypothesis, one has

$$\begin{aligned} \chi[m, n] &\sim -\chi[m-2\bar{p}, n+\bar{p}] , \\ \chi[m-1, n+1] &\sim -\chi[m-1-2(\bar{p}-1), n+1+(\bar{p}-1)] , \\ \chi[m, n-1] &\sim -\chi[m-2\bar{p}, n-1-\bar{p}] . \end{aligned} \tag{12.17}$$

Therefore, by adding eqs. (12.15) and (12.16) and by using (12.17), one finds

$$\chi[m+1, n] \sim -\chi[m+1-2(\bar{p}+1), n+(\bar{p}+1)] . \tag{12.18}$$

Eq. (12.18) shows that eq. (12.12) holds also for $p=\bar{p}+1$. □

Rule 3. *Let us consider the element* $\chi[m, n]$. *Suppose that, for some nonnegative integer* p, $\chi[m, n-p]$ *belongs to* $\mathscr{A}$ *and is represented by a point on a horizontal line; then*

$$\chi[m, n] \sim -\chi[m+p, n-2p] . \tag{12.19}$$

Proof. The proof of eq. (12.19) is based on the same algebraic manipulations used in the proof of Rules 1 and 2. □

The equivalence relations described by Rules 1, 2 and 3 are shown in Fig. 12.1. These rules must be used to connect points of the physical region ($m \geq 0$ and $n \geq 0$) of the square lattice. The points of the physical region of the lattice describe all the irreducible representations $\{(m, n)\}$ of SU(3) which label the elements $\{\chi[m, n]\}$ of the standard basis of $\mathscr{T}$.

For fixed integer $k \geq 3$, let us consider a generic element $\chi[m, n]$. If $\chi[m, n]$ belongs to $\mathscr{A}$, then $\chi[m, n] \sim 0$. If $\chi[m, n] \notin \mathscr{A}$, by using Rules 1, 2 and 3 recursively, it is easy to see that $\chi[m, n]$ is physically equivalent (with a well determined sign) to an element $\chi[a, b]$ represented by a point in the fundamental domain Δ_k. The points of Δ_k have coordinates (a, b) characterized by

$$\Delta_k \equiv \{(a, b)\} \text{ with } \begin{cases} 0 \leq a < k-2 , \\ 0 \leq b < -a+k-2 . \end{cases} \tag{12.20}$$

We shall denote by $\Psi[a, b]$ the class associated with the irreducible representation (a, b) of SU(3) with the point $(a, b) \in \Delta_k$. A generic element χ of $\mathscr{T}$ admits a linear decomposition in terms of the elements of the standard basis of $\mathscr{T}$

$$\chi = \sum_{m,n} \xi(m, n)\chi[m, n] . \tag{12.21}$$

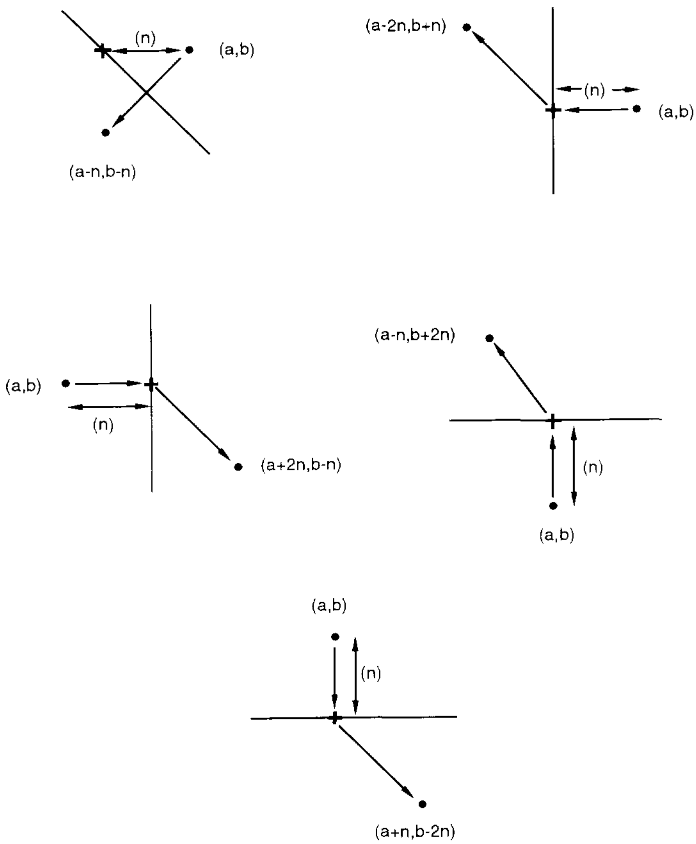

Fig. 12.1. Graphical representation of Rules 1, 2, 3.

Let Ψ be the class of physical equivalent elements associated with χ. Since the class corresponding to each $\chi[m, n]$ is the trivial class or a class $\Psi[a, b]$ with $(a, b) \in \Delta_k$, it follows from eq. (12.21) that

$$\Psi = \sum_{(a,b)\in\Delta k} \xi'(a, b)\Psi[a, b] , \tag{12.22}$$

where $\{\xi'(a, b)\}$ are linear combinations of the coefficients $\{\xi(a, b)\}$.

To sum up, assuming that for $k \geq 3$ each element of $I_{(k)}$ has the form (12.4), eq. (12.22) shows that the elements of $\mathscr{T}$ modulo the ideal $I_{(k)}$ admits a linear decomposition in terms of the classes $\{\Psi[a, b]\}$ with $(a, b) \in \Delta_k$. Therefore, $\{\Psi[a, b]\}$ form a basis of $\mathscr{T}_{(k)}$ that we call the standard basis. It remains to be verified that $\mathscr{T}_{(k)}$, defined above, is physically irreducible. This means that if the element $\Psi_* \in \mathscr{T}_{(k)}$ is physically equivalent to zero $\Psi_* \sim 0$, then $\Psi_* = 0$. The proof is very simple. Let us consider the Hopf link in which one component has colour Ψ_* and the other component has colour given by a generic element Ψ of $\mathscr{T}_{(k)}$. If $\Psi_* \sim 0$, the associated expectation value vanishes

$$\langle W(C_1;\Psi_*)W(C_2;\Psi)\rangle|_{S^3}=0\ . \tag{12.23}$$

Let us expand Ψ_* as in (12.22),

$$\Psi_*=\sum_{(m,n)\in\Delta_k}\xi_*(m,n)\Psi[m,n]\ . \tag{12.24}$$

Since eq. (12.23) holds for arbitrary Ψ, eq. (12.23) implies that, for any $(a, b)\in\Delta_k$, one has

$$\sum_{(m,n)\in\Delta_k}\xi_*(m,n)H[(m,n)\ (a,b)]=0\ . \tag{12.25}$$

The complex numbers $\{H[(m,n),(a,b)]\}$, for $(m,n)\in\Delta_k$ and $(a,b)\in\Delta_k$, can be understood as the matrix elements of a symmetric matrix called the Hopf matrix H. As shown in the Appendix, the Hopf matrix H is invertible; therefore, eq. (12.25) implies that

$$\xi_*(m,n)=0,\quad \text{for any}\,(m,n)\in\Delta_k\ . \tag{12.26}$$

This shows that $\Psi_*=0$. In conclusion, the results of this section are summarized by the following theorem.

Theorem 5. *For fixed integer $k\geq 3$, the reduced tensor algebra $\mathcal{T}_{(k)}$ coincides with the classes of elements of $\mathcal{T}$ modulo the ideal $I_{(k)}$ generated by the non-trivial null vectors ζ_1 and ζ_2 defined in Section* 11.

Finally we note that, as a linear space, $\mathcal{T}_{(k)}$ has a dimension which is equal to the number of representative points on the square lattice which belong to the fundamental domain Δ_k. Consequently, for fixed $k\geq 3$, the reduced tensor algebra $\mathcal{T}_{(k)}$ is of order $(k-1)(k-2)/2$. When $k=3$, the order of $\mathcal{T}_{(3)}$ is equal to unity; in this case, the fundamental domain Δ_3 contains a single point with coordinates $(0, 0)$. This means that, for $k=3$, there is only one class of physically equivalent states; this class can be represented by $\chi[0, 0]$.

13. Reduced tensor algebra for $k=1$

In this section we compute the reduced tensor algebra for $k=1$. Since $E_0[m,n]$ does not vanish when $k=1$ (see eq. (10.2)), in the construction of the reduced tensor algebra $\mathcal{T}_{(1)}$ one finds a situation which is quite different from the case $k\geq 3$. Clearly, when $k=1$ we cannot use the same argument that we presented in the previous section; nevertheless, we will show that the expectation value of a generic link L has a beautiful periodicity in the colour state space $\mathcal{T}$.

In order to find $\mathcal{T}_{(1)}$, we shall produce the explicit expression of $\langle W(L)\rangle|_{S^3}$

for a generic link L when $k=1$. As we shall show, we only need to consider the double crossings between two lines of a link diagram. These crossings can be analyzed by using the graphical rules introduced in [2,4]. Let us consider a part of a link diagram (tangle) which describes a two-string configuration. The no-crossing tangle, representing two parallel lines, can be decomposed as shown in Fig. 13.1. The "projectors" appearing on the right-hand side of the relation shown in Fig. 13.1 represent skein modules [2,4]. One can always imagine that the part of the link which is described by the tangle is contained inside a solid torus. Each projector simply selects one colour state of the standard basis of $\mathscr{T}$ which flows along the core of this solid torus. The tangle corresponding to a double crossing can be decomposed [2] as shown in Fig. 13.2, where the twist variable $\alpha(a, b)$ is given by

$$\alpha(\rho)=\alpha(a,b)=q^{Q(a,b)} . \tag{13.1}$$

When $k=1$, one has

$$\alpha(a,b)=\mathrm{e}^{-(2\pi i/3)(a^2+b^2+3a+3b+ab)} . \tag{13.2}$$

The algebraic structure of expression (13.2) shows that $\alpha(a, b)$ depends on the values of a and b modulo 3. Let us introduce the triality t_ρ of an irreducible representation $\rho\simeq(a, b)$, defined as

$$t_\rho=a-b \pmod 3) . \tag{13.3}$$

In our convention, the possible values taken by t_ρ are $\{-1, 0, 1\}$. The value of α variable is

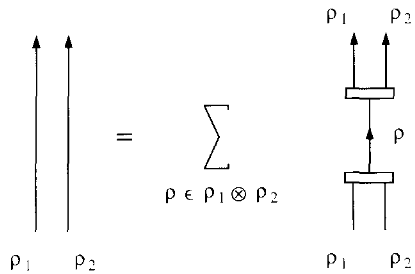

Fig. 13.1. Projection decomposition of the no-crossing tangle.

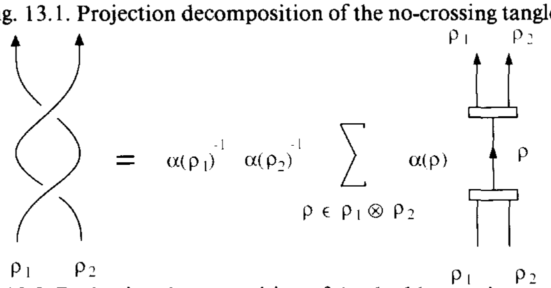

Fig. 13.2. Projection decomposition of the double crossing tangle.

$$\alpha(\rho)=e^{-2\pi i/3} \quad \text{if } t_\rho \neq 0\,, \qquad \alpha(\rho)=1 \quad \text{if } t_\rho = 0\,. \tag{13.4}$$

Triality is conserved in the sense that all the irreducible representations $\{\rho_i\}$ of SU(3), which enter the decomposition of the tensor product $\rho_1 \otimes \rho_2$, have the same triality:

$$\rho_1 \otimes \rho_2 = \bigoplus_i \rho_i\,, \qquad t_{\rho_i} = t_{\rho_1} + t_{\rho_2} \pmod 3 \quad \forall i\,. \tag{13.5}$$

Eqs. (13.4) and (13.5) imply that, for $k=1$, the decomposition illustrated in Fig. 13.2 takes the simple form shown in Fig. 13.3. The complex coefficient $Y_{\rho_1\rho_2}$ is given by

$$Y_{\rho_1\rho_2} = e^{(2\pi i/3) t_{\rho_1} t_{\rho_2}}\,. \tag{13.6}$$

Double undercrossing can be obtained simply by replacing $Y_{\rho_1\rho_2}$ with its complex conjugate $Y^*_{\rho_1\rho_2}$.

Let us now compute the expectation value of a Wilson line operator associated to a generic link L with n coloured components; the colour of the component C_i is given by the irreducible representation ρ_i. The relation shown in Fig. 13.3 permits us to transform L into a collection of n disjoint knots. Consequently, one finds

$$\langle W(C_1, ..., C_n; \rho_1, ..., \rho_n)\rangle|_{S^3} = F(t_{\rho_1}, ..., t_{\rho_n}) \langle W(C_1;\rho_1)\rangle|_{S^3} \cdots \langle W(C_n;\rho_n)\rangle|_{S^3}\,, \tag{13.7}$$

where

$$F(t_{\rho_1}, ..., t_{\rho_n}) = \exp\left(\frac{2\pi i}{3} \sum_{i<j} \mathrm{lk}(C_i, C_j) t_{\rho_i} t_{\rho_j}\right)\,. \tag{13.8}$$

At this stage, by using the relation of Fig. 13.3 and eq. (13.4), we can transform each knot C_i into the unknot. Therefore we have

$$\langle W(C_1, ..., C_n; \rho_1, ..., \rho_n)\rangle|_{S^3} = G(t_{\rho_1}, ..., t_{\rho_n}) \left(\prod_{i=1}^n E_0[\rho_i]\right)\,, \tag{13.9}$$

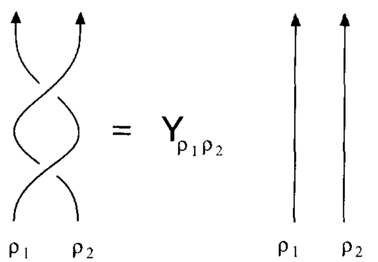

Fig. 13.3. Equivalence between tangles for $k=1$.

with

$$G=\exp\left[\frac{-2\pi i}{3}\left(\sum_{i<j} 2\,\mathrm{lk}(C_i, C_j)t_{p_i}t_{p_j}+\sum_i \mathrm{lk}(C_i, C_{if})t_{p_i}^2\right)\right]. \tag{13.10}$$

At this point, from eq. (13.9) and eq. (13.10) it is clear that the equivalence classes of physically equivalent colour states are characterized only by triality. Therefore, the reduced tensor algebra $\mathcal{T}_{(1)}$ is of order equal to three [16]. The elements of the standard basis of $\mathcal{T}_{(1)}$ are denoted by $\{\Psi[0], \Psi[1], \Psi[-1]\}$; the structure constants are determined by triality conservation,

$$\begin{aligned}&\Psi[0]\Psi[0]=\Psi[0], \quad \Psi[0]\Psi[1]=\Psi[1], \quad \Psi[0]\Psi[-1]=\Psi[-1],\\ &\Psi[1]\Psi[1]=\Psi[-1], \quad \Psi[1]\Psi[-1]=\Psi[0],\\ &\Psi[-1]\Psi[-1]=\Psi[1].\end{aligned} \tag{13.11}$$

Since, for $k=1$, one has $E_0[a, b]=D(a, b)$, each element of $\mathcal{T}$ corresponds to an element of $\mathcal{T}_{(1)}$ according to

$$\chi[a, b] \Rightarrow \begin{cases} D(a, b)\Psi[0] & \text{if } a-b=0 \pmod 3, \\ D(a, b)\Psi[1] & \text{if } a-b=1 \pmod 3, \\ D(a, b)\Psi[-1] & \text{if } a-b=-1 \pmod 3. \end{cases} \tag{13.12}$$

As a check, let us verify that the Hopf matrix, defined in terms of basis elements of $\mathcal{T}_{(1)}$, is nonsingular. We consider the Hopf link, shown in Fig. 8.1, in which each component is characterized by an element of the standard basis $\{\Psi[0], \Psi[1], \Psi[-1]\}$ of $\mathcal{T}_{(1)}$. The expectation value of the associated Wilson line operator is

$$H[\Psi[i], \Psi[j]]=H_{ij}, \tag{13.13}$$

where, according to eqs. (13.9) and (13.10), the 3×3 matrix H is given by

$$H=\begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \end{pmatrix}. \tag{13.14}$$

Since the Hopf matrix H is invertible, the set $\mathcal{T}_{(1)}$ is physically irreducible; this concludes the construction of the reduced tensor algebra for $k=1$.

14. Reduced tensor algebra for $k=2$

When $k=2$, the value of the unknot is

$$E_0[m, n]\,|_{k=2}=\begin{cases} 0 & \text{for } m \text{ and } n \text{ odd;} \\ -(n+1)/2 & \text{for } n \text{ odd and } m \text{ even;} \\ -(m+1)/2 & \text{for } m \text{ and odd and } n \text{ even;} \\ -(m+n+2)/2 & \text{for } m \text{ and } n \text{ even.} \end{cases} \tag{14.1}$$

Property 2 and eq. (14.1) imply that each irreducible representation $\chi[m, n]$, with m and n odd, is physically equivalent to the null element of $\mathscr{T}_{(2)}$. The set of representations $\{[m, n]\}$ with m and n odd is called $\mathscr{A}_2$. The value of the twist variable for $k=2$ is

$$\alpha(a, b)=(-1)^{a+b+ab}\, e^{-(i\pi/3)(a-b)^2} . \qquad (14.2)$$

Apart from the irreducible representations physically equivalent to the null element, we find that the value of $\alpha(a, b)$ depends only on triality. More precisely, if the irreducible representation ρ does not belong to $\mathscr{A}_2$, we have

$$\begin{aligned} &\alpha(\rho)=e^{2\pi i/3} && \text{if } t_\rho \neq 0 , \\ &\alpha(\rho)=1 && \text{if } t_\rho = 0 . \end{aligned} \qquad (14.3)$$

In analyzing the double crossings, we can ignore the representations contained in $\mathscr{A}_2$; indeed, each projector on a representation of this kind gives a vanishing contribution. Therefore, eq. (14.3) implies that the relation shown in Fig. 13.3 is valid also for $k=2$; we only need to compute the new values of the coefficients $Y_{\rho_1\rho_2}$. From eq. (14.2) and the relation shown in Fig. 13.2, for $k=2$ one finds

$$Y_{\rho_1\rho_2}=e^{-(2\pi i/3)t_{\rho_1}t_{\rho_2}} . \qquad (14.4)$$

The same argument that we used in the previous section now gives

$$\langle W(C_1, ..., C_n; \rho_1, ..., \rho_n)\rangle|_{S^3}=G'(t_{\rho_1}, ..., t_{\rho_n})\left(\prod_{i=1}^{n} E_0[\rho_i]\right) , \qquad (14.5)$$

with

$$G'=\exp\left[\frac{2\pi i}{3}\left(\sum_{i<j} 2\, \mathrm{lk}(C_i, C_j)t_{\rho_i}t_{\rho_j}+\sum_i \mathrm{lk}(C_i, C_{if})t_{\rho i}^2\right)\right] . \qquad (14.6)$$

The equivalence classes of physically equivalent colour states are again characterized only by triality. Therefore, the reduced tensor algebra $\mathscr{T}_{(2)}$ is of order equal to three [16]. The elements of the standard basis of $\mathscr{T}_{(2)}$ are denoted by $\{\Psi[0], \Psi[1], \Psi[-1]\}$ with structure constants

$$\begin{aligned} &\Psi[0]\Psi[0]=\Psi[0], \quad \Psi[0]\Psi[1]=\Psi[1], \quad \Psi[0]\Psi[-1]=\Psi[-1] , \\ &\Psi[1]\Psi[1]=\Psi[-1], \quad \Psi[1]\Psi[-1]=\Psi[0] , \\ &\Psi[-1]\Psi[-1]=\Psi[1] . \end{aligned} \qquad (14.7)$$

Each element of $\mathscr{T}$ corresponds to an element of $\mathscr{T}_{(2)}$ according to

$$\chi[a, b] \Rightarrow \begin{cases} E_0[a, b]\Psi[0] & \text{if } a-b=0 \text{ (mod 3)} , \\ E_0[a, b]\Psi[1] & \text{if } a-b=1 \text{ (mod 3)} , \\ E_0[a, b]\Psi[-1] & \text{if } a-b=-1 \text{ (mod 3)} , \end{cases} \qquad (14.8)$$

where $E_0[a, b]$ is given in eq. (14.1).

When $k=2$ the Hopf matrix H, which is defined in terms of basis elements $\{\Psi[0], \Psi[1], \Psi[-1]\}$ of $\mathscr{T}_{(2)}$, is given by

$$H=\begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{-2\pi i/3} & e^{2\pi i/3} \\ 1 & e^{2\pi i/3} & e^{-2\pi i/3} \end{pmatrix}. \tag{14.9}$$

Clearly, the Hopf matrix H is invertible and then $\mathscr{T}_{(2)}$ is physically irreducible.

We note that the reduced tensor algebras $\mathscr{T}_{(1)}$ and $\mathscr{T}_{(2)}$ are isomorphic and coincide with the group algebra of the center Z_3 of SU(3).

15. Conclusions

The CS expectation values of the Wilson line operators represent ambient isotopy invariants of oriented framed coloured links. The colour state space associated to each link component coincides with the tensor algebra $\mathscr{T}$ which is the complex algebra induced by the representation ring of the gauge group. For links with m components, these invariants are linear functions on $\mathscr{T}^{\otimes m}$. In this article we have shown how to compute these invariants for links defined in $\mathbb{R}^3$ and S^3 when the gauge group is SU(3). For integer values of the coupling constant k, different elements of $\mathscr{T}$ not necessarily correspond to different values of the observables. This leads to the introduction of the reduced tensor algebra $\mathscr{T}_{(k)}$ whose elements correspond to the set of physically inequivalent colour states. We have given the complete classification of the reduced tensor algebras $\{\mathscr{T}_{(k)}\}$ associated with SU(3). These algebras are of finite order and play a crucial role in the surgery presentation of three-manifolds [7].

As suggested by Witten in Ref. [1], the braiding properties of the analytic conformal blocks of the WZNW model in two dimensions are strictly connected with the skein relations satisfied by the expectation values of the CS theory. This implies that the algebra of the fusion rules of the $SU(3)_l$ WZNW conformal theory is isomorphic with the reduced tensor algebra $\mathscr{T}_{(k)}$ with $k=l+3$. This topic will be discussed in Ref. [7].

Appendix A

For fixed integer $k\geq 3$, let us consider the values $\{H[(m, n), (a, b)]\}$ of the Hopf link for $(m, n)\in\Delta_k$ and $(a, b)\in\Delta_k$. These complex numbers can be understood as the matrix elements of a symmetric matrix called the Hopf matrix H. The elements of the standard basis of $\mathscr{T}_{(k)}$ are $\{\Psi[a, b]\}$ with $(a, b)\in\Delta_k$; to simplify the notation, we shall denote them simply by $\{\psi_i\}$ with the collective index i

running from 1 to the dimension of $\mathcal{T}_{(k)}$. The element ψ_1 corresponds to $\Psi[0, 0]$ and, if ψ_i represents $\Psi[a, b]$, then ψ_{i*} denotes $\Psi[b, a]$. The matrix elements of H are

$$H_{ij} = H[\psi_i, \psi_j] \,. \tag{A.1}$$

We wish to prove that H is invertible.

First of all, let us evaluate H^2. By definition, one has

$$(H^2)_{ij} = \sum_l H_{il} H_{lj} \,. \tag{A.2}$$

By taking into account that the values of the Hopf link are given in eq. (8.4), one finds that expression (A.2) consists of a combination of geometric finite sums. The direct computation of these geometric finite sums gives [16]

$$(H^2)_{ij} = \left[\frac{3k^2}{256 \sin^6(\pi/k) \cos^2(\pi/k)} \right] \delta_{ij*} \,. \tag{A.3}$$

This equation shows that the Hopf matrix H is invertible; indeed,

$$(H^{-1})_{ij} = \sum_l \left[\frac{3k^2}{256 \sin^6(\pi/k) \cos^2(\pi/k)} \right]^{-1} H_{il} \delta_{l*j} \,. \tag{A.4}$$

References

[1] E. Witten, Quantum field theory and the Jones polynomial, Commun. Math. Phys. 121 (1989) 351–399.

[2] E. Guadagnini, The universal link polynomial, Int. J. Mod. Phys. A 7 (1992) 877–945.

[3] E. Guadagnini, Surgery rules in quantum Chern–Simons field theory, Nucl. Phys. B 375 (1992) 381–398;
E. Guadagnini and S. Panicucci, Topological field theory and surgery on three-manifolds, Nucl. Phys. B 388 (1992) 159–226.

[4] E. Guadagnini, *The Link Invariants of the Chern–Simons Field Theory, De Gruyter Expositions in Mathematics,* Vol. 10 (de Gruyter, Berlin, 1993).

[5] N.Y. Reshetikhin and V.G. Turaev, Invariants of 3-manifolds via link polynomials and quantum groups, Invent. Math. 103 (1991) 547–597;
V.G. Turaev, *Quantum Invariants of 3-Manifolds* (de Gruyter, Berlin), to be published.

[6] W.B.R. Lickorish, Three-manifold invariants from the combinatorics of the Jones polynomial, Pacific J. Math. 149 (1991) 337–347;
R. Kirby and P. Melvin, Evaluations of the 3-manifold invariants of Witten and Reshetikhin–Turaev for sl(2), Invent. Math. 105 (1991) 473–545;
H.R. Morton and P.M. Strickland, Satellites and surgery invariants, in: *Knots 90,* ed. A. Kawauchi (de Gruyter, Berlin, 1992);
T. Kohno, Topological invariants for 3-manifolds using representations of mapping class groups I, Topology (1992).

[7] E. Gauadagnini and L. Pilo, SU(3) Chern–Simons field theory in three-manifolds, J. Geom. Phys. (1994), to be published.

[8] E. Guadagnini, M. Martellini and M. Mintchev, Wilson lines in Chern–Simons theory and link invariants, Nucl. Phys. B 330 (1990) 575–607.

[9] C. Becchi, A. Rouet and R. Stora, Renormalization of the Abelian Higgs–Kibble model, Commun. Phys. 42 (1975) 127–162; Renormalization of gauge theories, Ann. Phys. (NY) 98 (1976) 287–328.
[10] F. Delduc, O. Piguet, C. Lucchesi and S.P. Sorella, Exact scale invariance of the Chern–Simons theory in the Landau gauge, Nucl. Phys. B 346 (1990) 313–328.
[11] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, Singapore, 1980).
[12] A.M. Polyakov, Fermi–Bose transmutations induced by gauge fields, Mod. Phys. Lett. A 3 (1988) 325–328.
[13] E. Guadagnini, M. Martellini and M. Mintchev, Braids and quantum group symmetry in Chern–Simons theory, Nucl. Phys. B 336 (1990) 581–609.
[14] E. Guadagnini, Chern–Simons theory and fusion algebras, Phys. Lett. B 260 (1991) 353–358.
[15] W.B.R. Lickorish and K.C. Millett, A polynomial of oriented links, Topology 26 (1987) 107–141.
[16] L. Pilo, Teoria di Chern–Simons con gruppo di gauge SU(3) ed invarianti di 3-varietà, Thesis (1993).
[17] D. Rolfsen, *Knots and Links* (Publish or Perish, 1976).